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Put, Marius van der; Voskuil, Harm H.

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Discontinuous subgroups of $\mathrm{PGL}_2(K)$

Marius van der Put * and Harm H. Voskuil

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1. Introduction

Let K be an algebraically closed field, which is complete with respect to a non-Archimedean valuation. A finitely generated discontinuous subgroup $\Gamma \subset \mathrm{PGL}_2(K)$ has, acting on $\mathbf{P}^1(K)$, a compact set of limit points $\mathcal{L}$. Its set of ordinary points $\Omega = \mathbf{P}^1(K) \setminus \mathcal{L}$ is a rigid analytic subspace of $\mathbf{P}^1(K)$ on which Γ acts discontinuously. The quotient Ω/Γ is known to be a smooth, complete, irreducible algebraic curve over K. In this paper we investigate the groups Γ such that $\Omega/\Gamma \cong \mathbf{P}^1_K$. Let $\mathrm{pr}: \Omega \to \mathbf{P}^1(K)$ denote the induced morphism of rigid spaces. A point $y \in \Omega$ will be called *ramified* if its stabilizer $\{\gamma \in \Gamma \mid \gamma(y) = y\}$ is non-trivial. A *branch point for* Γ is a point $x \in \mathbf{P}^1(K)$ such that $x = \mathrm{pr}(y)$ for some ramified point y. The group Γ has many normal subgroups Δ of finite index, such that Δ is a free group. The pair (Δ, Γ) induces a (ramified) Galois covering $X := \Omega/\Delta \to \Omega/\Gamma = \mathbf{P}^1_K$ of the projective line with X a Mumford curve. Such a covering is called a Mumford covering (see also [7]) and every Mumford covering of the projective line is obtained by a pair (Δ, Γ). The branch points of the covering $X \to \mathbf{P}^1_K$ induced by (Δ, Γ) are the branch points of Γ.

The *aim of this paper* is to classify all possible groups Γ as amalgams of certain finite trees of groups (see 3.12, 3.14, 4.10, 4.11) and to give a formula (see 5.3) for *the number of branch points* $\mathrm{br}(\Gamma)$ of Γ. The results depend heavily on the characteristic p_K of K and the characteristic p_k of the residue field k of K. This classification turns out to be extraordinary rich and complicated. There are exceptional groups Γ in case $p_K = 0$ and $p_k = 2, 3$ or 5, for which no reasonable classification seems to exist. We exclude these groups and restrict ourselves to *ordinary* groups (see 4.1 and 4.2).

A central role is played by subtrees of the (generalized) Bruhat–Tits tree of $\mathrm{PGL}_2(K)$, which consists of the lattice classes in K^2. The finite subgroups G of $\mathrm{PGL}_2(K)$ are investigated and one associates to G a finite tree $\mathrm{Tree}(G)$ which captures the configuration of the ramification points of $\mathbf{P}^1_K \to \mathbf{P}^1_K/G$ (see Sections 2.4 and 2.5). Using these trees, all

* Corresponding author.
E-mail address: mvdput@math.rug (M. van der Put).

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Γ of the form $G_1 *_{G_3} G_2$ are classified (see 3.5 and 3.9). This part of the paper extends and completes earlier work of F. Herrlich [5].

The main purpose of Section 4 is to construct the tree $\mathcal{T}^c$ associated to an indecomposable, ordinary group Γ and to show that the finite tree of groups $\mathcal{T}^c/\Gamma$ has the correct properties. In Section 5, this finite tree of groups is used for the formula for $\mathrm{br}(\Gamma)$.

Examples and lists of Mumford coverings of $\mathbf{P}^1_K$, unramified outside $\{0, 1, \infty\}$ and for a field K of characteristic 0, have been given by Y. André [1] and F. Kato [6]. For these examples the corresponding groups Γ satisfy $\mathrm{br}(\Gamma) = 3$. These groups are exceptional (see 5.5) and there is no overlap between [1,6] and the present paper. In [2] a list of Mumford coverings of $\mathbf{P}^1_K$ with two branch points is given for fields K of positive characteristic. In principle this list can be deduced from our classification of the trees $\mathcal{T}^c/\Gamma$ having $\mathrm{br}(\Gamma) = 2$ (see 5.8). The methods of [2] however, are quite different from the ones developed here.

2. Trees and finite subgroups of $\mathrm{PGL}_2(K)$

In this section the material on subtrees of the (generalized) Bruhat–Tits tree for the group $\mathrm{PGL}_2(K)$ is presented. More information and more detailed proofs can be found in [4] and [3]. The information on the action of finite subgroups on $\mathbf{P}^1_K$, needed in the sequel, is also provided in this section. We note that some of this material is already present in [2,5,6].

2.1. Lattices and trees

The valuation ring of K is denoted by K^0 and its maximal ideal by K^{00}. The characteristic of K is denoted by p_K and that of its residue field $k = K^0/K^{00}$ by p_k. The field K is supposed to be algebraically complete and we suppose that $p_k > 0$ ($p_k = 0$ seems uninteresting). Any reduced algebraic variety over K (or over k) will be identified with its set of K-valued (or k-valued) points. A *lattice* $M \subset K^2$ is a free submodule over K^0 of rank two. Two lattices M_1, M_2 are called equivalent if there exists a $\lambda \in K^*$ with $M_1 = \lambda M_2$. The equivalence class of the lattice M will be denoted by $[M]$. As usual $\mathbf{P}^1(K)$ is identified with $\mathbf{P}(K^2)$. For a given lattice M one has a *reduction map*:

$$\mathrm{red}_{[M]} : \mathbf{P}^1(K) = \mathbf{P}\big(K^2\big) = \mathbf{P}(M) \to \mathbf{P}(M \otimes k) = \mathbf{P}^1(k).$$

This reduction map depends only on the class of M. For three distinct points $Kv_1, Kv_2, Kv_3 \in \mathbf{P}^1(K)$ there is a unique lattice class $[M]$ such that the three points are mapped to three distinct points of $\mathbf{P}(M \otimes k) = \mathbf{P}^1(k)$. One can describe $[M]$ explicitly by the following. Let $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 = 0$ be the unique (up to a multiple) non-trivial linear relation between v_1, v_2, v_3. Then M is the K^0-module generated by $\lambda_1 v_1, \lambda_2 v_2, \lambda_3 v_3$.

For a lattice class $[M_1] \neq [M]$ one defines $\mathrm{red}_{[M]}([M_1]) \in \mathbf{P}(M \otimes k)$ as follows. We may suppose that $M = K^0 e_1 + K^0 e_2$ and $M_1 = K^0 e_1 + K^0 \pi e_2$ for suitable e_1, e_2 and $0 < |\pi| < 1$. Then $\mathrm{red}_{[M]}([M_1]) := \mathrm{red}_{[M]}(K e_1)$.

For a moment we replace the field K be a local field L with residue field ℓ and generator π_L of the maximal ideal of L^0. The collection $\mathcal{BT}(L)$ of all lattice classes in L^2 is a locally finite tree, called the Bruhat–Tits building of $\mathbf{P}^1(L)$ or of $\mathrm{PGL}_2(L)$, defined by

(a) The vertices of the tree are the lattice classes.
(b) $\{[M_1],[M_2]\}$ is an edge if one can choose the lattices M_1, M_2 such that $M_1 \supset M_2 \supset \pi_L M_1$ and $M_2 \neq M_1, \pi_L M_1$.

Every vertex is contained in precisely $1 + \#\ell$ edges. The ends of $\mathcal{BT}(L)$ are in a 1–1 correspondence with the points of $\mathbf{P}^1(L)$.

We return now to the algebraically closed and complete valued field K and write $\mathcal{BT}$, the "*generalized Bruhat–Tits tree*," for the *collection of all classes of lattices*. This object is too large to be a (locally finite) tree but still has a *tree-like structure* in the following sense. Let $[M_1] \neq [M_2]$ denote two classes of lattices. One can represent them by lattices $M_1 \supset M_2$ such that M_1/M_2 is isomorphic to $K^0/(\pi)$ for some π with $0 < |\pi| < 1$. This representation is unique up to multiplication by some $\lambda \in K^*$. The *segment* $[[M_1],[M_2]]$, also denoted by $\mathrm{conv}([M_1],[M_2])$, joining $[M_1]$, $[M_2]$ consists of all $[M]$ such that $M_1 \supset M \supset M_2$. For any three distinct classes of lattices v_1, v_2, v_3 there exists a unique lattice class v_4 such that precisely one of the following statements holds:

(a) $v_4 = v_i$ for some $i \in \{1,2,3\}$ and $v_i \in [v_j, v_k]$ where $\{i,j,k\} = \{1,2,3\}$.
(b) $v_4 \neq v_1, v_2, v_3$ and for all $i \neq j \in \{1,2,3\}$ one has $[v_i, v_4] \cap [v_j, v_4] = \{v_4\}$.

This last property implies that the collection of all lattice classes does not contain a cycle. We will consider certain subsets of $\mathcal{BT}$ which are actually locally finite trees.

(1) *The tree $\mathcal{T}_F$ of a finite subset F of $\mathbf{P}^1(K)$.*

For any finite set $F \subset \mathbf{P}^1(K)$ of cardinality $\geqslant 3$. The *vertices* of $\mathcal{T}_F$ are the lattice classes $[M]$ such that the image $\mathrm{red}_{[M]}(F)$ consists of at least three points. For two vertices $[M_1] \neq [M_2]$ one considers the map $\mathrm{red}_{[M_1],[M_2]} := \mathrm{red}_{[M_1]} \times \mathrm{red}_{[M_2]} : \mathbf{P}^1(K) \to \mathbf{P}(M_1 \otimes k) \times \mathbf{P}(M_2 \otimes k)$. The image can be seen to be the union of the two lines $\mathbf{P}^1(k) \times \{a_2\}$ and $\{a_1\} \times \mathbf{P}^1(k)$ intersecting at (a_1, a_2). The pair $\{[M_1],[M_2]\}$ is called an *edge* if there is no point $f \in F$ with $\mathrm{red}_{[M_1],[M_2]} f = (a_1, a_2)$. This defines a graph which is actually a finite tree. One combines all reductions maps for the vertices $[M]$ of $\mathcal{T}_F$ to a map $\mathrm{red}_F = \prod_{[M]} \mathrm{red}_{[M]} : \mathbf{P}^1(K) \to \prod_{[M]} \mathbf{P}(M \otimes k)$. The image of red_F is called $\overline{(\mathbf{P}^1, F)}$ and has the following properties:

(a) $\overline{(\mathbf{P}^1, F)}$ is a reduced variety over k. Each irreducible component is a $\mathbf{P}^1(k)$. Each singular point is a normal intersection of two irreducible components.
(b) The dual graph of $\overline{(\mathbf{P}^1, F)}$ is defined as follows. Its set of vertices are the irreducible components and its set of edges are the singular points. The dual graph is the tree $\mathcal{T}_F$.
(c) The map $\mathrm{red}_F : F \to \mathrm{red}_F(F)$ is bijective and $R_F(F)$ consists of non singular points.

(d) Every irreducible component has at least three special points, i.e., intersections with other components or elements in $\mathrm{red}_F(F)$.

(2) *The tree $\mathcal{T}_F$ of a infinite subset F of $\mathbf{P}^1(K)$, having compact closure $\overline{F}$.*

The vertices and the edges of tree $\mathcal{T}_F$ are defined as in (1) above. Let F^* denote the (compact) subset of the non-isolated points of $\overline{F}$. Put $\Omega = \mathbf{P}^1(K) \setminus F^*$. One can combine the reductions with respect to all vertices $[M]$ to a reduction map $\mathrm{red}_F : \Omega \to \overline{(\Omega, F)}$, where the latter is the direct limit of the $\overline{(\mathbf{P}^1, G)}$ taken over all finite subsets G of F. One can prove that $\overline{(\Omega, F)}$ is a locally finite reduced variety over k satisfying the above properties (a)–(d), where one has to replace in (c) and (d) the set F by $F \cap \Omega$. The dual graph of $\overline{(\Omega, F)}$ is the locally finite tree $\mathcal{T}_F$. The points of $\overline{F}$ outside Ω are not mapped by red_F to points in $\overline{(\Omega, F)}$. These points correspond with the ends of $\mathcal{T}_F$. For more information and proofs, see [4].

(3) *Discrete subsets $\mathcal{M}$ of $\mathcal{BT}$ and the tree $\mathcal{T}_{\mathcal{M}}$.*

A set $\mathcal{M}$ of lattice classes will be called *discrete* if there exists a compact $F \subset \mathbf{P}^1(K)$ such that for all $[M] \in \mathcal{M}$ the set $\mathrm{red}_{[M]}(F)$ contains at least three elements. In other words $\mathcal{M}$ is a subset of the vertices of $\mathcal{T}_F$ for a suitable compact F. A (*locally finite*) *tree* T in $\mathcal{BT}$ is defined by

(a) A finite or discrete subset V of $\mathcal{BT}$.
(b) An edge of T is a pair $v_1, v_2 \in V$, $v_1 \neq v_2$ such that $[v_1, v_2] \cap V = \{v_1, v_2\}$.
(c) There are no cycles in the graph T.

From the discreteness of V one concludes that every $v \in V$ is contained in only finitely many edges. Moreover property (c) can be seen to be equivalent to

(c′) For any three distinct elements $v_1, v_2, v_3 \in V$, such that $v_i \notin [v_j, v_k]$ if $\{i, j, k\} = \{1, 2, 3\}$, the unique lattice class $[M]$ such that the $R_{[M]}(v_i)$, $i = 1, 2, 3$, are distinct, belongs to V.

It is not difficult to show that for any tree T in $\mathcal{BT}$ there exists a compact set $F \subset \mathbf{P}^1(K)$ such that $T = \mathcal{T}_F$. We note that this set F is far from unique. Its set of non-isolated points F^* is unique and will be called the set of limit points of T. In particular one can associate to T a reduction $\mathrm{red}_T : \Omega \to \overline{(\Omega, T)}$ with $\Omega := \mathbf{P}^1(K) \setminus F^*$ and $\overline{(\Omega, T)} := \overline{(\Omega, F)}$ having the properties of (2) above.

In general a discrete set of lattice classes $\mathcal{M}$ is not the set of vertices of a tree. One defines the tree $\mathcal{T}_{\mathcal{M}}$ to be the smallest tree (contained in $\mathcal{BT}$) such that $\mathcal{M}$ is contained in the set of its vertices. One constructs $\mathcal{T}_{\mathcal{M}}$ as follows.

The convex hull $\mathrm{conv}(\mathcal{M})$ is defined as the union $\bigcup \mathrm{conv}([M_1], [M_2])$ taken over all $[M_1], [M_2] \in \mathcal{M}$. The *interior* of $\mathrm{conv}(\mathcal{M})$ consists of the $[M] \in \mathrm{conv}(\mathcal{M})$ for which there exists a pair of equivalence classes $[M_1], [M_2] \in \mathrm{conv}(\mathcal{M})$, $[M_1], [M_2] \neq [M]$, such that $[M] \in \mathrm{conv}([M_1], [M_2])$. The set of lattice classes $V(\mathrm{conv}(\mathcal{M})) \subset \mathcal{BT}$ consist of:

(i) The $[M] \in \mathrm{conv}(\mathcal{M})$ that are not contained in the interior of $\mathrm{conv}(\mathcal{M})$.
(ii) The $[M] \in \mathrm{conv}(\mathcal{M})$ for which there exist $[M_1], [M_2], [M_3] \in \mathrm{conv}(\mathcal{M})$ that have the following two properties:
   (a) $[M]$ is contained in the interior of both $\mathrm{conv}([M_1], [M_2])$ and $\mathrm{conv}([M_1], [M_3])$.
   (b) $\mathrm{conv}([M_1], [M_2]) \cap \mathrm{conv}([M_1], [M_3]) = \mathrm{conv}([M_1], [M])$.

The discrete set $V(\mathrm{conv}(\mathcal{M}))$ already defines a locally finite tree in $\mathcal{BT}$. The discrete set $\mathcal{M} \cup V(\mathrm{conv}(\mathcal{M}))$ defines a locally finite tree in $\mathcal{BT}$ which is easily seen to be $\mathcal{T}_{\mathcal{M}}$.

Consider any $[M] \in \mathcal{BT}$. Then we define a map $\mathrm{red}_{[M]}$ from the edges $e \in \mathcal{T}_{\mathcal{M}\cup\{[M]\}}$ that contain $[M]$ to $\mathbf{P}(M \otimes k)$ by $\mathrm{red}_{[M]}(e) = \mathrm{red}_{[M]}([M'])$ if e is the edge with vertices $[M]$, $[M']$. If $[M]$ happens to be a vertex v of $\mathcal{T}_{\mathcal{M}}$ then we will often write red_v for the map $\mathrm{red}_{[M]}$.

(4) *The affinoid covering of Ω corresponding to a tree T.*

As in (3) above we consider an infinite tree T in $\mathcal{BT}$ with set of limit points $\mathcal{L}$ and put $\Omega = \mathbf{P}^1(K) \setminus \mathcal{L}$. Let $\mathrm{red}_T : \Omega \to \overline{(\Omega, T)}$ denote the reduction. A vertex v of T corresponds to an irreducible component, say L_v, of $\overline{(\Omega, T)}$. Let L_v^* be obtained from L_v by omitting the double points and define $X_v := \mathrm{red}_T^{-1}(L_v^*)$. Then X_v is an affinoid subset of $\mathbf{P}^1(K)$ with canonical reduction L_v^*. For an edge e of T there is a corresponding double point $d \in \overline{(\Omega, T)}$ which is the intersection of two irreducible components, say L_{v_1}, L_{v_2}, where v_1, v_2 are the vertices of the edge e. Let $(L_{v_1} \cup L_{v_2})^*$ denote the union of L_{v_1} and L_{v_2} where one has omitted all double points different from d. One defines $X_e := \mathrm{red}_T^{-1}(L_{v_1} \cup L_{v_2})^*$. This is an affinoid subset of Ω. If the two vertices v_1, v_2 are not end vertices, then $(L_{v_1} \cup L_{v_2})^*$ is the canonical reduction of the affinoid X_e. The affinoid covering of Ω associated to T is the admissible affinoid covering $\{X_v, X_e \mid \text{all } v, e\}$. In the case that T has no extremal vertices, $\{X_e \mid \text{all } e\}$ is an admissible pure (or formal) affinoid covering of Ω and this covering defines again the reduction $\overline{(\Omega, T)}$ of Ω.

2.2. The tree of a finite subgroup G of $\mathrm{PGL}_2(K)$ *and separating lattices*

Suppose that the set of ramification points $\mathcal{F}$ for the action of G on $\mathbf{P}^1(K)$ has cardinality at least 3. Then the tree $\mathrm{Tree}(G)$ of G is defined to be the tree of $\mathcal{F}$. The group G acts on $\overline{(\mathbf{P}^1(K), \mathcal{F})}$ and on $\mathrm{Tree}(G)$. The quotient graph $\mathrm{Tree}(G)/G$ is again a tree and there is a subtree $\mathrm{Tree}(G)^*$ of $\mathcal{T}_G$ which is mapped bijectively to the quotient tree. We make $\mathrm{Tree}(G)^*$ into a tree of groups by attaching to each vertex its stabilizer in G. The stabilizer of any edge is clearly the intersection of the stabilizers of its two vertices. The tree $\mathrm{Tree}(G)$ is completely described by the tree of groups $\mathrm{Tree}(G)^*$. In the sequel we will give a list of the finite subgroups G and their associated trees of groups. The set of ramification points $\mathcal{F}$ is the union of at most three G-orbits (as we will see). In the pictures for the tree of groups we will indicate the position of the images of these orbits on $\overline{(\mathbf{P}^1(K), \mathcal{F})}$.

Let $[M]$ be a G-invariant lattice. The action of G on $[M]$ induces an action of G on $\mathbf{P}(M \otimes k) = \mathbf{P}^1(k)$. A G-invariant lattice M is called *separating* if any two points of $\mathcal{F}$

belonging to distinct G-orbits are mapped to distinct points of $\mathbf{P}^1(k)$. This property has the following translation. Let $\mathbf{P}^1(K) \to \mathbf{P}^1(K)/G$ denote the quotient map. The images of the G-orbits of $\mathcal{F}$ are the branch points. There is also a quotient map $\mathbf{P}^1(k) \to \mathbf{P}^1(k)/G$ and an induced map $\phi : \mathbf{P}^1(K)/G \to \mathbf{P}^1(k)/G$. Then $[M]$ separates if ϕ is injective on the set of branch points. The questions that we want to answer are

(i) Is there only one class of invariant lattices?
(ii) Is there a separating lattice? If so, is its class unique?

The answers depend heavily on the group G and the characteristics p_K, p_k. In the sequel we have to treat each case separately. In the calculations we will, without explicitly stating this, replace $G \subset \mathrm{PGL}(2, K) = \mathrm{PSL}(2, K)$ by its preimage $\widetilde{G} \subset \mathrm{SL}(2, K)$ and identify G with $\widetilde{G}/\{\pm 1\}$. The group $\widetilde{G}$ actually acts on K^2 and can be represented by some matrices. An element of G is thus represented by a matrix modulo ± 1.

2.3. *Trees for $\Gamma \subset \mathrm{PGL}_2(K)$ and indecomposable groups*

As before, $\Gamma \subset \mathrm{PGL}_2(K)$ denotes an infinite discontinuous group, which is finitely generated and satisfies $\Omega/\Gamma \cong \mathbf{P}^1(K)$. Let $\mathcal{L} \subset \mathbf{P}^1(K)$ denote the set of the limits points of Γ. By definition $\Omega = \mathbf{P}^1(K) \setminus \mathcal{L}$. There are three possibilities:

(i) *$\mathcal{L}$ consists of one point.*

We may suppose that $\mathcal{L} = \{\infty\}$. It can be seen that in this case $p = p_K > 0$ and that the group has the form $\{z \mapsto \zeta_m^i z + a \mid i = 0, \ldots, m-1;\ a \in A\}$ where p does not divide m and ζ_m is a primitive mth-root of unity. Let $\mathbf{F}_q$ denote the smallest finite field containing ζ_m. Then A is a $\mathbf{F}_q$-linear subspace of K. Moreover, for every positive real number R the set $\{a \in A \mid |a| \leqslant R\}$ is a finite dimensional $\mathbf{F}_q$-linear space. We note that Γ is not finitely generated and this example will play a minor role in the sequel.

(ii) *$\mathcal{L}$ consists of two points.*

We may suppose that $\mathcal{L} = \{0, \infty\}$. It can be seen that Γ is conjugated to a group consists of the transformations $\{z \mapsto \zeta_m^i \pi^n z^{\pm 1} \mid i = 0, \ldots, m-1;\ n \in \mathbf{Z}\}$, where ζ_m is a primitive mth-root of unity and $\pi \in K$ satisfies $0 < |\pi| < 1$. For $m > 1$, the group Γ is the amalgam $D_m *_{C_m} D_m$ and for $m = 1$ one has $\Gamma = C_2 * C_2$. Let $\mathcal{F}$ denote the set of the fixed points of Γ and let $\mathcal{T}$ denote the corresponding tree of the reduction $\overline{(K^*, \mathcal{F})}$. Then $\mathcal{T}/\Gamma$ has two vertices and one edge. Let $\mathcal{T}/\Gamma^*$ be a subtree of $\mathcal{T}$ which maps bijectively to $\mathcal{T}/\Gamma$. One makes $\mathcal{T}/\Gamma^*$ into a tree of groups by assigning to each vertex and edge its stabilizer in Γ. By [8] the amalgam of this tree of groups is Γ. This is consistent with the above description of Γ as an amalgam. Finally, we note that *the number of branch points of the map $\Omega = K^* \to \Omega/\Gamma \cong \mathbf{P}^1(K)$ is equal to 4 if $p_K \neq 2$ and equal to 2 if $p_K = 2$.*

(iii) *$\mathcal{L}$ is infinite, compact and has no isolated points.*

We are mainly interested in this case. The infinite set $\mathcal{L}$ determines a reduction of Ω and a tree $\mathcal{T}_{\mathcal{L}}$ on which Γ acts faithfully. An *inversion* is defined as an edge e with vertices v_1, v_2 such that some element in Γ permutes v_1, v_2. If Γ acts without inversion on $\mathcal{T}_{\mathcal{L}}$ then one defines $\mathcal{T} := \mathcal{T}_{\mathcal{L}}$. If Γ acts with inversions then $\mathcal{T}$ is defined as the tree obtained from $\mathcal{T}_{\mathcal{L}}$ by subdividing each edge where an inversion occurs. In general one can also enlarge $\mathcal{L}$ to F by adding to $\mathcal{L}$ a finite set of Γ orbits of points of Ω. The resulting tree $\mathcal{T}_F$ can be obtained from $\mathcal{T}_{\mathcal{L}}$ by adding finitely many Γ-orbits of vertices and edges. Again Γ acts on this new tree. The tree $\mathcal{T}$ can be obtained as a $\mathcal{T}_F$ for a suitable F.

The quotient $\mathcal{T}/\Gamma$ is a finite graph and in fact it is a tree since $\Omega/\Gamma \cong \mathbf{P}^1(K)$. There exists a finite subtree $\mathcal{T}/\Gamma^*$ of $\mathcal{T}$ which maps bijectively to $\mathcal{T}/\Gamma$. We make $\mathcal{T}/\Gamma^*$ into a tree of groups by assigning to each vertex and edge its stabilizer subgroup in Γ. According to [8], the group Γ is equal to the amalgam of $\mathcal{T}/\Gamma^*$. Moreover, the tree of groups $\mathcal{T}/\Gamma^*$ does not depend on the choice of $\mathcal{T}$ in an essential way (indeed, one tree is obtained from the other by a sequence of subdivisions and contractions).

The group Γ and the tree of groups are called *indecomposable* if every vertex and every edge of $\mathcal{T}/\Gamma^*$ has a non-trivial stabilizer.

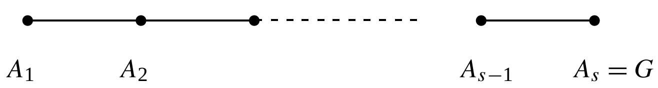
In the general situation, one removes from $\mathcal{T}/\Gamma^*$ the vertices and the edges with trivial stabilizers. There results a number of indecomposable trees of groups $\mathcal{T}/\Gamma_i^*$, $i = 1, \ldots, s$. Each piece has an amalgam Γ_i and Γ is the free product of the indecomposable subgroups Γ_i. The group Γ_i is again a discontinuous, finitely generated subgroup of $\mathrm{PGL}_2(K)$, has a set of limits points $\mathcal{L}_i \subset \mathcal{L}$. We note that Γ_i can be a finite group. Put $\Omega_i = \mathbf{P}^1_K \setminus \mathcal{L}_i$. Then Ω_i/Γ_i is isomorphic to $\mathbf{P}^1(K)$ since the graph $\mathcal{T}/\Gamma_i^*$ is a tree. This decomposition of Γ as a free product of indecomposable groups is helpful for the study of the maximal finite subgroups of Γ and their intersections. Indeed, according to [8], each finite subgroup of $\Gamma_1 * \cdots * \Gamma_s$ is conjugated to a subgroup of Γ_i for a unique i. Similarly, let $G_1, G_2 \subset \Gamma$ be two finite groups with $G_1 \cap G_2 \neq 1$. Then there is a unique i and a $\gamma \in \Gamma$ such that $\gamma G_j \gamma^{-1} \subset \Gamma_i$ for $j = 1, 2$.

2.4. Finite subgroups of $\mathrm{PGL}_2(K)$ *for* $p_K = p > 0$

In the sequel $G \neq 1$ is a finite subgroup of $\mathrm{PGL}_2(K)$. We investigate various properties of G and the morphism $\pi : \mathbf{P}^1(K) \to \mathbf{P}^1(K)/G \cong \mathbf{P}^1(K)$.

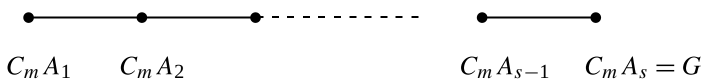
Suppose that G is a p-group. Let $h \in G$ be an element of order p in the center of G. Then h has a single fixed point which we may suppose to be ∞. For any $g \in G$ one has $hg(\infty) = gh(\infty) = g(\infty)$ and thus $g(\infty) = \infty$. Thus G is contained in the group $\{\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\}$ and can be identified with a finite subgroup of K. The map π has only one ramification point and only one branch point. The contribution of the ramification point for the Riemann–Hurwitz formula is $2(-1 + \#G)$.

Since there is only one ramification point one cannot make the reduction of $\mathbf{P}^1_K$ w.r.t. this set. The group itself is a finite subgroup A of K and consists of the maps $\{z \mapsto z + a \mid a \in A\}$. One considers a set $S \subset \mathbf{P}^1_K$ consisting of one orbit and the point ∞. For this set one can make the reduction of $\mathbf{P}^1_K$, its dual graph. The latter is divided by the action of A and yields a the following graph of groups, which is similar to the one of the next case

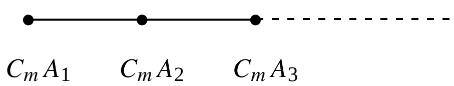


Suppose that G lies in a Borel subgroup, in which case we will call G a group of Borel type. We exclude the two cases: G is a p-group and G is a cyclic group. Let $A \subset G$ be the unique p-Sylov subgroup of G and let m be the order of the cyclic group G/A. Then G is the semi-direct product of C_m and A. Then $A \subset$ is a finite-dimensional vector space over $\mathbf{F}_{p^r} = \mathbf{F}_p(\xi)$ where ξ is a primitive mth-root of unity. The group G can be represented by the set of matrices $\{\left(\begin{smallmatrix} c & a \\ 0 & 1 \end{smallmatrix}\right) \mid c \in \langle\xi\rangle \subset \mathbf{F}_{p^r}^*,\ a \in A\}$. The set of ramification points is $A \cup \{\infty\}$. The stabilizer of each $a \in A$ is a cyclic group of order m. The stabilizer of ∞ is G. The map π has two branched points, namely $\pi(\infty)$ and $\pi(0) = \pi(A)$. The contribution of the point ∞ for the Riemann–Hurwitz–Zeuthen formula is $(m+1)p^n - 2$ where $p^n = \#A$. The group described above will be called *type* $B(n,m)$ where n is the dimension of the $\mathbf{F}_p$-vector space A. Clearly m divides $p^r - 1$ and $p^n - 1$. Moreover all groups of type $B(n,m)$ are isomorphic (in general not by conjugation in $\mathrm{PGL}_2(K)$).

One can form the reduction of $\mathbf{P}^1$ with respect to the set of ramification points $A \cup \{\infty\}$ and its dual graph. The latter is divided out by the action of G. The result is a graph of groups. The groups attached to the vertices and the edges are the stabilizer subgroups of these objects. The set of absolute values $\{|a| \mid a \in A,\ a \neq 0\}$ is written as $v_1 < v_2 < \cdots < v_s$. Let $A_i = \{a \in A \mid |a| \leqslant v_i\}$. Then A_i is a $\mathbf{F}_q$-subspace of A. The picture is the following:



We remark that a similar analysis and picture can be made for an infinite discontinuous subgroup G of a Borel group (see 2.3 case (i)). This will only be used for discontinuous groups which are not finitely generated (see 5.9). The group G can be represented by the matrices $\{\left(\begin{smallmatrix} c & a \\ 0 & 1 \end{smallmatrix}\right) \mid c \in \langle\xi\rangle \subset \mathbf{F}_{p^r}^*,\ a \in A\}$, where ξ is a primitive mth-root of unity and $\mathbf{F}_{p^r} = \mathbf{F}_p(\xi)$. Further A is an infinite discrete $\mathbf{F}_{p^r}$-subspace of K. Let $v_1 < v_2 < v_3 < \cdots$ denote the absolute values of the non zero elements of A and let $A_i := \{a \in A_i \mid |a| \leqslant v_i\}$. Then each A_i is a finite dimensional $\mathbf{F}_{p^r}$-vector space. The action of G on $\mathbf{A}^1$ is discontinuous; the set of the ramification points is A. The graph of groups obtained, similarly as above, has the following picture



Suppose that G does not lie in a Borel subgroup, in other words G is not of Borel type.

Proposition 2.1. *Suppose that G is not of Borel type. Then*

(1) π *has at most three branch points.*

(2) *π has two branch points if p divides $\#G$.* (*Moreover, one branch point is wildly ramified and the other is tamely ramified.*)
(3) *π has three branch points if p does not divide $\#G$.*
(4) *Suppose that π has three branch points. Then G is one of the groups D_n, A_4, S_4, A_5, provided that p does not divide the order of this group.*

Proof. Let $a_1, \ldots, a_s$ denote the branch points of π. The ramification index of a_i is written as $e_i p^{d_i}$ with e_i is not divisible by p. The contribution of the points of $\mathbf{P}^1$ above a_i to the Riemann–Hurwitz–Zeuthen formula is

$$\frac{\#G}{e_i p^{d_i}}\big((e_i+1)p^{d_i}-2\big).$$

This formula yields

$$2-\frac{2}{\#G}=\sum_{i=1}^{s}\frac{(e_i+1)p^{d_i}-2}{e_i p^{d_i}}.$$

The term $\frac{(e_i+1)p^{d_i}-2}{e_i p^{d_i}}$ is $\geqslant 1$ if $d_i \neq 0$ and is $\geqslant 1/2$ if $d_i = 0$. Moreover, if p divides the order of G, then the fixed point of an element of order p in $\mathbf{P}^1$ is mapped to some a_i and thus $d_i \geqslant 1$. This proves (1)–(3).

From the formula

$$2-\frac{2}{\#G}=\sum_{i=1}^{3}\frac{e_i-1}{e_i}$$

one derives, as in the characteristic zero situation, the possibilities for e_1, e_2, e_3 and the structure of the group G. □

Remarks. In [9] there is a list of the groups described in part (2) of Proposition 5.4, namely,

(a) $p=2$, group D_n, $(n,2)=1$. The group can be given by $z \mapsto \zeta_n^a z^{\pm 1}$ where ζ_n is a primitive nth-root of unity and $a = 0, \ldots, n-1$. The ramification points are: $0, \infty$ for the elements $z \mapsto \zeta_n^a z$ and $\{\zeta_n^a \mid a = 0, \ldots, n-1\}$ for the elements of order two in D_n. The group D_n can also be realized as the subgroup of $\mathrm{PSL}_2(\mathbf{F}_{2^s})$, generated by the matrices $\left(\begin{smallmatrix}\alpha+1 & 1\\ \alpha & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$. Here s is minimal such that $\zeta_n \in \mathbf{F}_{2^{s+1}}$ and $\alpha = \zeta_n + \zeta_n^{-1}$.
(b) $p=3$ and group A_5. This group can be realized as the subgroup of $\mathrm{PSL}_2(\mathbf{F}_9)$ which contains $\mathrm{PSL}_2(\mathbf{F}_3)$.
(c) The group $\mathrm{PGL}(2, \mathbf{F}_q)$. The ramification points are $\mathbf{P}(\mathbf{F}_{q^2})$. There are two orbits namely $\mathbf{P}(\mathbf{F}_q)$ and $\mathbf{P}(\mathbf{F}_{q^2}) \setminus \mathbf{P}^1(\mathbf{F}_q)$. The stabilizer of a point in the first orbit is a Borel subgroup. The stabilizer of a point in the second orbit is a cyclic subgroup of order $q+1$.
(d) The group $\mathrm{PSL}(2, \mathbf{F}_q)$. The situation is similar. The ramification points are $\mathbf{P}(\mathbf{F}_{q^2})$. There are two orbits, namely, $\mathbf{P}(\mathbf{F}_q)$ and $\mathbf{P}(\mathbf{F}_{q^2}) \setminus \mathbf{P}^1(\mathbf{F}_q)$. The stabilizer of a point

in the first orbit is a Borel subgroup. The stabilizer of a point in the second orbit is a cyclic subgroup of order $(q+1)/2$.

Proposition 2.2. *Suppose that G is not of Borel type. The G-invariant lattices M are all equivalent. The reduction map*

$$\mathrm{red}_{[M]} : \mathbf{P}^1(K) = \mathbf{P}(M) \to \mathbf{P}(M \otimes k) = \mathbf{P}^1(k)$$

with respect to M, induces an injective homomorphism $G \to \mathrm{PGL}_2(k)$. Moreover, $\mathrm{red}_{[M]}$ induces a bijective map from the ramification points of G on $\mathbf{P}^1(K)$ to the ramification points of the action of G on $\mathbf{P}^1(k)$. For any ramification point $x \in \mathbf{P}^1(K)$, the stabilizer in G of x coincides with the stabilizer in G of $\mathrm{red}_{[M]}(x)$. In particular, the lattice M separates the branch points.

Proof. Suppose that G has two non-equivalent lattices M_1, M_2. Then one may suppose that $M_1 = K^0 e_1 + K^0 e_2 \supset M_2 = K^0 e_1 + K^0 \rho e_2$ for some element ρ with $0 < |\rho| < 1$. For $g \in G$ one considers the reduction modulo the maximal ideal K^{00} of K^0 of the matrix of g w.r.t. the basis e_1, e_2. This matrix can be represented by $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ with $a \in k^*$ and $b \in k$. The homomorphism $G \to k^*/\{\pm 1\}$ has as image an m-cyclic group with $p \nmid m$ and its kernel N is a p-group. Then G is a semi-direct product of a cyclic group C_m of order m and N. The group N has a single ramification point. This is also the fixed point of the elements in C_m. Thus G is of Borel type. This proves the first statement.

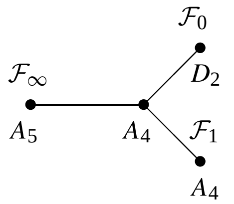
The kernel N of $G \to \mathrm{PGL}_2(k)$ consists of the $g \in G$ such that g acts as the identity on $M \otimes k$. Thus $g = 1 + a$ where the linear map a maps M into ρM for some ρ with $0 < |\rho| < 1$. Then, since the group G is finite, for a suitable q, a power of p, one has $g^q = 1$. Thus N is a normal p-group. If $N \neq 1$, then all elements of N have a single fix point, say ∞. Since N is normal, one has $g(\infty) = \infty$ for all $g \in G$. This contradicts the assumption that G is not of Borel type. The last part of the proposition follows from Proposition 2.1. $\square$

2.5. Finite subgroups of $\mathrm{PGL}_2(K)$ *with* $p_K = 0$

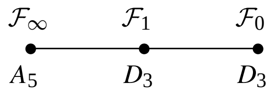
The groups G are D_n, A_4, S_4, A_5. The set of ramification points $\mathcal{F}$ has three orbits $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_\infty$ and correspond with the fibres above $0, 1, \infty \in \mathbf{P}^1/G = \mathbf{P}^1$. The ramification indices of $0, 1, \infty$ are denoted by e_0, e_1, e_∞. The above groups correspond to the following triples (e_0, e_1, e_∞):

$$(2, 2, n); \quad (2, 3, 3); \quad (2, 3, 4); \quad (2, 3, 5).$$

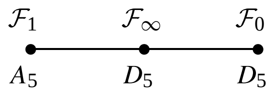
For each case there is a reduction of $\mathbf{P}^1_K$ with respect to the set of ramification points $\mathcal{F}$. Its dual graph is divided by the action of the group and this produces a graph (tree) of groups. If p_k does not divide the order of the group then this graph is just one point with stabilizer the group. The pictures for the interesting cases, where p_k divides the order of the group are the following:



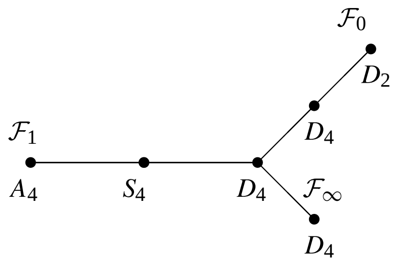
A_5 and $p_k = 2$.
Unique invariant lattice, not separating.



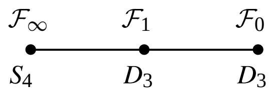
A_5 and $p_k = 3$.
Unique invariant lattice, not separating.



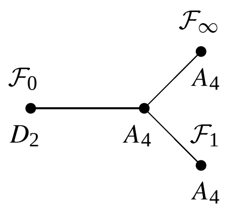
A_5 and $p_k = 5$.
Unique invariant lattice, not separating.



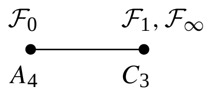
S_4 and $p_k = 2$.
Unique invariant lattice, not separating.



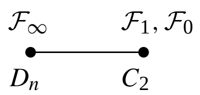
S_4 and $p_k = 3$.
Unique invariant lattice, not separating.



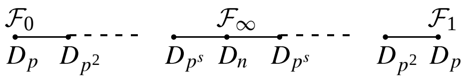
A_4 and $p_k = 2$.
Unique separating lattice.
More invariant lattices.



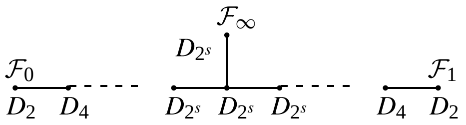
A_4 and $p_k = 3$.
Unique invariant lattice, not separating.



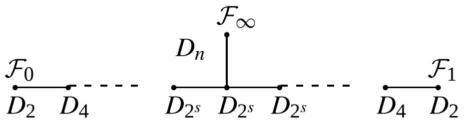
D_n, n odd, $p_k = 2$.
Unique invariant lattice, not separating.



D_n; $p_k > 2$; $n = p_k^s m$; $(m, p_k) = 1$.
Unique separating lattice.
For $m = 1$ more invariant lattices.



D_{2^s} and $p_k = 2$.
Unique separating lattice.
More invariant lattices.



D_n; $p_k = 2$; $n = 2^s m$; $(m, 2) = 1$; $m > 1$.
Unique invariant lattice, not separating.

Comments on the calculations

(1) Suppose that there are at least two classes of invariant lattices for the finite (non-cyclic) group $G \subset \mathrm{PGL}_2(K)$. Then there are invariant lattices

$$M_1 = K^0 e_1 + K^0 e_2 \supset M_2 = K^0 e_1 + K^0 \pi e_2$$

for some $\pi \in K$ with $0 < |\pi| < 1$. This induces a homomorphism $\psi : G \to k^*/\{\pm 1\}$ given by $\psi(g)$ satisfies $g(e_1) \equiv \psi(g) e_1$ modulo $K^{00} M_1$. For any element g in the kernel of ψ one has that g^{p_k} acts trivially on $M_1 \otimes k$. This implies that the order of g is some power of p_k and that the kernel of ψ is a p_k-group and contains the commutator subgroup $[G, G]$. The image $\psi(G)$ is a cyclic group of order prime to p_k. From this one easily derives the only possibilities for the pairs (G, p_k), namely $(D_{p_k^s}, p_k)$ for $p_k \geqslant 2$ and $(A_4, p_k = 2)$.

(2) For each of the above groups G one needs an explicit representation of G acting on $\mathbf{P}^1(K)$. Using this, one calculates $\mathcal{F}$ and the reduction $\overline{(\mathbf{P}^1(K), \mathcal{F})}$ and the tree of groups is derived from the latter.

3. Realizable amalgams

Let $\Gamma \subset \mathrm{PGL}_2(K)$ be a finitely generated, infinite, discontinuous group such that $\Omega/\Gamma \cong \mathbf{P}^1(K)$. We are investigating the structure of Γ and the number of branch points $\mathrm{br}(\Gamma)$ of $\Omega \to \Omega/\Gamma \cong \mathbf{P}^1(K)$. According to 2.3, Γ is a free product of indecomposable groups Γ_i, $i = 1, \ldots, s$. We start by proving that $\mathrm{br}(\Gamma) = \sum \mathrm{br}(\Gamma_i)$. If $\mathcal{L}$ consists of two points then $\mathrm{br}(\Gamma) = 4$ if $p_K \neq 2$ and is equal to 2 if $p_K = 2$ (see 2.3). In the sequel we will suppose that $\mathcal{L}$ has more than two points. As in 2.3, one considers the subdivision $\mathcal{T}$ of the tree $\mathcal{T}_{\mathcal{L}}$ on which the group Γ acts without inversions. Let T_Γ denote a chosen embedding of $\mathcal{T}/\Gamma$ in $\mathcal{T}$. This makes of T_Γ a tree of groups. Let T'_Γ denote the subset of T_Γ consisting of the vertices and edges which have a non-trivial stabilizer. Then T'_Γ is the disjoint union of finite trees $T_1, \ldots, T_s$. Let Γ_i denote the subgroup of Γ generated by the stabilizers of the edges and vertices of T_i. Then Γ_i is an indecomposable group and $\Gamma = \Gamma_1 * \cdots * \Gamma_s$. Let $\mathcal{L}_i$ denote the set of the limit points of Γ_i. Then $\Omega_i := \mathbf{P}^1(K) \setminus \mathcal{L}_i$ is the set of ordinary points of Γ_i and $\Omega_i/\Gamma_i \cong \mathbf{P}^1_K$. The group Γ_i can be finite, in which case $\Omega_i = \mathbf{P}^1(K)$. Since $\Gamma_i \subset \Gamma$ one has $\mathcal{L}_i \subset \mathcal{L}$ and therefore $\Omega \subset \Omega_i$.

Lemma 3.1. *Any point $x \in \Omega_i$, which is fixed by a non-trivial finite subgroup of Γ_i belongs to Ω.*

Proof. Let $G_x \subseteq \Gamma_i$ be the stabilizer of the point $x \in \Omega_i$. Since G_x is finite, the group G_x stabilizes a vertex $v \in \mathcal{T}$. After replacing x by $\gamma(x)$ for a suitable element $\gamma \in \Gamma_i$, we may assume that $v \in T_i \subseteq T_\Gamma \subset \mathcal{T}$.

If x is a limit point of Γ, then x determines a unique end of $\mathcal{T}$. Let $L \subset \mathcal{T}$ be the halfline that starts in the vertex v and that corresponds to the end x of $\mathcal{T}$. Clearly, G_x stabilizes the halfline L.

Let $\mathcal{T}_i \subseteq \mathcal{T}$ be the subtree, whose edges and vertices are stabilized by non-trivial finite subgroups of Γ_i. Then $\mathcal{T}_i/\Gamma_i = T_i$. The ends of $\mathcal{T}_i$ correspond to the limit points of Γ_i. In particular, $L \subset \mathcal{T}_i$ and x is a limit point of Γ_i. This contradicts $x \in \Omega_i$. □

Proposition 3.2. *With the above notations one has* $\mathrm{br}(\Gamma) = \sum_{i=1}^{s} \mathrm{br}(\Gamma_i)$.

Proof. Let $x \in \Omega$ represent a branch point for Γ. The stabilizer G_x in Γ is then finite and non-trivial. There is a unique $i \in \{1, \ldots, s\}$ such that $\gamma G_x \gamma^{-1}$ lies in Γ_i for a suitable $\gamma \in \Gamma$. Then $\gamma(x)$ represents the same branch point of Γ and represents moreover a branch point for Γ_i since $\Omega \subset \Omega_i$. On the other hand a branch point for Γ_i is represented by a point $x \in \Omega_i$ and is also a branch point for Γ according to Lemma 3.1. □

According to 3.2, the questions of this paper are reduced to indecomposable groups. We will make these problems more precise. A *finite indecomposable tree of groups* (T, G) is a finite tree T with the additional structure:

(a) for the vertices v and edges e there are associated finite, non-trivial, finite groups G_v, G_e which can be realized as subgroups of $\mathrm{PGL}_2(K)$,
(b) for every edge e with vertices v_1, v_2 there are given injective homomorphisms $G_e \to G_{v_1}$ and $G_e \to G_{v_2}$. In the sequel we will just identify G_e with a subgroup of G_{v_1} and G_{v_2}.

The group of (T, G), i.e., the amalgam of the tree of groups (T, G), will be denoted by Γ. We recall from [8], that there is an *abstract tree* $\mathrm{Tree}(T, G)$ on which Γ acts. Its defining properties are:

T is a subtree of $\mathrm{Tree}(T, G)$, and for every vertex v or edge e of T, the groups G_v, G_e are the stabilizers in Γ of v and e. Moreover, the map $T \to \mathrm{Tree}(T, G)/\Gamma$ is an isomorphism of trees.

An *embedding* τ *of* (T, G) *in* $\mathcal{BT}$ is a map τ from the vertices of T to lattice classes, and for every vertex v an injective homomorphism $\tau_v : G_v \to \mathrm{PGL}_2(K)$ such that:

(a) The subtree of $\mathcal{BT}$ generated by all $\tau(v)$ is isomorphic to T.
(b) $\tau_v(G_v)$ stabilizes the lattice class $\tau(v)$.
(c) For any edge $e = \{v_1, v_2\}$ the restrictions of τ_{v_1} and τ_{v_2} to G_e coincide.

The embedding τ is called a *realization* if the induced homomorphism from the amalgam Γ of (T, G) to $\mathrm{PGL}_2(K)$ is injective and moreover its image is a discontinuous subgroup of $\mathrm{PGL}_2(K)$. For convenience we will identify Γ with its image in $\mathrm{PGL}_2(K)$.

Let a realization τ of (T, G) be given. For a vertex v of T, the stabilizer F in Γ of $\tau(v)$ is equal to $\tau_v(G_v)$. Indeed, $F \supset \tau_v(G_v)$ and $\tau_v(G_v)$ is a maximal finite subgroup of Γ. The map $\tau : T \to \mathcal{BT}$ extends uniquely to a Γ-equivariant map τ^* from the vertices of $\mathrm{Tree}(T, G)$ to $\mathcal{BT}$. This map is injective since any maximal finite subgroup of Γ is conjugated to a group G_v for a unique vertex v of T. We note that the subtree $\mathcal{T}$ of $\mathcal{BT}$, generated by the image of τ^*, has in general more vertices. Moreover, for an edge $e = \{v_1, v_2\}$ of $\mathrm{Tree}(T, G)$, the pair $\{\tau^* v_1, \tau^* v_2\}$ need not be an edge of $\mathcal{T}$. Indeed, according to part (3) of Section 2.1, new vertices and edges will occur if the image of τ^* contains three lattice classes $[M_i]$, $i = 1, 2, 3$, which do not lie on a segment of $\mathcal{BT}$ and such that the unique lattice class $[M]$ determined by $\{[M_i]\}_{i=1,2,3}$ does not lie in the image of τ^*. Nevertheless, Γ acts on $\mathcal{T}$ and this action has no inversions since Γ has no inversions on $\mathrm{Tree}(T, G)$. The quotient graph $\mathcal{T}/\Gamma$ can be seen to be a finite tree. An embedding of this quotient graph in $\mathcal{T}$ makes it into a finite tree of groups. The latter is essentially some subdivision of (T, G).

Problem 1. Classify the realizable finite indecomposable trees of groups.

Since a given tree of groups can be changed by subdivision or contraction, this question can only be handled if we introduce the notion of *contracted* tree of groups. This notion, which will be defined in 3.16, seems at first sight to give a restriction on the subgroups of $\mathrm{PGL}_2(K)$ under consideration.

Problem 2. What is $\mathrm{br}(\Gamma)$ for a realizable contracted (T, G)?

Problem 3. Is every finitely generated, discontinuous, indecomposable $\Gamma \subset \mathrm{PGL}_2(K)$ with $\Omega/\Gamma \cong \mathbf{P}^1_K$, isomorphic to the amalgam of a contracted, finite, indecomposable tree of groups (T, G)?

For the first question, F. Herrlich [5] has given several criteria. The condition (b) of Satz 1 of [5] can be formulated as follows: Let τ be an embedding of (T, G) in $\mathcal{BT}$. Then τ is a realization if for every edge $e = \{v_1, v_2\}$ and every $[M] \in [\tau(v_1), \tau(v_2)]$, $[M] \neq \tau(v_1), \tau(v_2)$ and every

$$g \in \bigcup_v \tau_v(G_v)$$

such that $g \notin \tau_{v_1}(G_e) = \tau_{v_2}(G_e)$ one has that $g([M])$ does not lie in the convex hull in $\mathcal{BT}$ spanned by all $\tau(v)$. We will reformulate this criterion and give an independent proof for the case of edges. The latter will lead to a classification of all groups $\Gamma := G_1 *_{G_3} G_2$, with $1 \neq G_3 \neq G_1, G_2$, which are realizable as a discontinuous subgroup of $\mathrm{PGL}_2(K)$. For a more general situation a criterion is formulated and proved which makes it possible to realize the contracted trees of groups by induction.

Theorem 3.3 (Herrlich's criterion).

(1) (The criterion for edges.) *Let* G_1, G_2 *be finite subgroups of* $\mathrm{PGL}_2(K)$ *and let* $G_3 \neq 1$ *be a proper subgroup of* G_1 *and* G_2*. The natural homomorphism* $\Gamma := G_1 *_{G_3} G_2 \to \mathrm{PGL}_2(K)$ *is a realization of* $G_1 *_{G_3} G_2$ *as a discontinuous subgroup if and only if*:
   (1a) *There are lattice classes* $[M_1] \neq [M_2]$ *such that* $[M_i]$ *is* G_i*-invariant for* $i = 1, 2$.
   (1b) *There is a lattice class* $T \in [[M_1], [M_2]]$ *having the property*: *if* $\alpha \in G_1 \cup G_2$ *satisfies* $\alpha T = T$*, then* $\alpha \in G_3$.

(2) (A more general case.) *Let a finite tree of groups* (T, G) *be given and let* $e = \{v_1, v_2\}$ *be an edge. Let* (T^1, G^1)*,* (T^2, G^2) *denote the trees of groups obtained by deleting the edge* e *and suppose* $v_1 \in T^1$*,* $v_2 \in T^2$*. An embedding* τ *of* (T, G) *is a realization if*:
   (2a) *The restriction of the embedding* τ *to each* (T^1, G^1)*,* (T^2, G^2) *is a realization. Let* Γ^1*,* Γ^2 *denote the resulting discontinuous subgroups of* $\mathrm{PGL}_2(K)$.
   (2b) *There is a lattice class* $V \in [\tau(v_1), \tau(v_2)]$ *with* $V \neq \tau(v_1), \tau(v_2)$*, such that for* $g_i \in \Gamma^i \setminus \tau(G_e)$*,* $i = 1, 2$*, one has that* $V \neq g_1 V, g_2 V$ *and* $V \in [g_1 V, g_2 V]$.

Observation 3.4 (*Invariant lattice classes*). Suppose first that $p_K = 0$. Let $A \in \mathrm{PGL}_2(K)$ be an element of order $m > 1$ with eigenvectors e_1 and e_2 in K^2. Then the lattice classes $[K^0 e_1 + K^0 \lambda e_2]$ with $\lambda \in K^*$ are invariant under A. We will call this infinite line the *axis* of A (or of the group generated by A). If m is not divisible by p_k, then there are no other invariant lattice classes. The same holds if m is divisible by p_k, but not equal to some power of p_k. For $m = p_k^s$ a lattice class $[M]$ is invariant if and only if there is a lattice class $[M_1]$ of the form $M_1 = K^0 e_1 + K^0 \lambda e_2$ such that for a suitable choice of M_2, representing $[M]$, one has $M_1/M_2 = K^0/\pi K^0$ with $|\pi| \geqslant |\zeta_{p_k^s} - 1|$. Here $\zeta_{p_k^s}$ denotes a primitive p_k^sth root of unity. In geometric terms, $[M]$ is invariant under A if and only if its "distance" to the axis of A is less than or equal to $-\log |\zeta_{p_k^s} - 1|$.

Suppose now that $p_K = p_k = p > 0$. Let $A \in \mathrm{PGL}_2(K)$ have finite order m, then either m is not divisible by p or $m = p$. In the first case A has two independent eigenvectors e_1, e_2 and the set of invariant lattice classes is again the axis of A. In the second case, there is a basis e_1, e_2 for which A has the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The set of the invariant lattice classes is $\{[K^0 e_1 + K^0(a e_1 + b e_2)] \mid a, b \in K;\ 0 < |b| \leqslant 1\}$. This set is a subtree of the tree of all lattice classes.

Proof. (1) Suppose that the amalgam is realizable. Then we consider invariant lattices $[M_1]$, $[M_2]$ for the groups G_1, G_2. Since $\Gamma = G_1 *_{G_3} G_2$ contains hyperbolic elements, one has that $[M_1] \neq [M_2]$. Take an element $\alpha \in G_1 \setminus G_3$, then the set of elements in $[[M_1], [M_2]]$ which are invariant under α has the form $[[M_1], [M_1(\alpha)]]$ for some $[M_1(\alpha)] \neq [M_2]$. Indeed, the group generated by α and G_2 contains a hyperbolic element and thus $[M_2]$ is not stable under α. Observation 3.4 proves the existence of $[M_1(\alpha)]$. Let $\alpha_1 \in G_1 \setminus G_3$ be such that the length of $[[M_1], [M_1(\alpha_1)]]$ is maximal. Similarly for $\beta \in G_2 \setminus G_3$ the set of β-invariant lattice classes in $[[M_1], [M_2]]$ is $[[M_2(\beta)], [M_2]]$ with $[M_2(\beta)] \neq [M_1]$. Let $\beta_1 \in G_2 \setminus G_3$ be such that the length of $[[M_2(\beta_1)], [M_2]]$ is maximal. If a lattice class S lies in the intersection of $[[M_1], [M_1(\alpha_1)]]$ and $[[M_2(\beta_1)], [M_2]]$ then S is invariant under $\alpha_1 \beta_1 \in \Gamma$. This element has infinite order, is therefore hyperbolic and has no invariant lattice class. So we conclude that the above intersection is empty and we take for T any element in $[[M_1(\alpha_1)], [M_2(\beta_1)]]$, different from the two endpoints of this segment. Clearly T has the required property.

Suppose that conditions (1a) and (1b) are satisfied. We make the following *observation*: If $g_i \in G_i \setminus G_3$, $i = 1, 2$, then $g_1T \neq g_2T$ and $T \in [g_1T, g_2T]$ (and, of course, $T \neq g_1T, g_2T$).

Indeed, the intersection $[[M_1], T] \cap [g_1[M_1], g_1T]$ is equal to $[[M_1], S_1]$ with $S_1 \in [[M_1], T]$ and $S_1 \neq T$. Similarly, $[T, [M_2]] \cap [g_2T, g_2[M_2]] = [S_2, [M_2]]$ with $S_2 \in [T, [M_2]]$ and $S_2 \neq T$. This proves the observation.

Consider a word $w_s \cdots w_1$ in $G_1 *_{G_3} G_2$, where $w_i \in G_1 \setminus G_3$ if i is odd and $w_i \in G_2 \setminus G_3$ if i is even. By induction on s, we will show that the sequence of lattice classes $T, w_sT, w_sw_{s-1}T, \ldots, w_sw_{s-1} \cdots w_1T$ are distinct consecutive points on the segment $[T, w_s \cdots w_1T]$.

The statement is obvious for $s = 1$. The induction hypothesis says that $T, w_{s-1}T, \ldots, w_{s-1} \cdots w_1T$ are consecutive points on the segment $[T, w_{s-1} \cdots w_1T]$. By the observation, $w_s^{-1}T, T, w_{s-1}T$ are consecutive points on the segment $[w_s^{-1}T, w_{s-1}T]$. Since the collection of all lattice classes is a treelike object, one finds that $w_s^{-1}T, T, w_{s-1}T, \ldots, w_{s-1} \cdots w_1T$ are consecutive points on the segment $[w_s^{-1}T, w_{s-1} \cdots w_1T]$. Applying w_s to the latter one obtains the statement for s. With the above notations one has that T lies in the segment $[[M_1(\alpha_1)], [M_2(\beta_1)]]$ and is not an end point. Therefore there is a $c > 0$, such that each segment $[T, w_sT], [T, w_{s-1}T], \ldots, [T, w_1T]$ has length $\geqslant c$. This implies that the distance between $w_s \cdots w_1T$ and T is $\geqslant sc$.

We note that a similar statement holds for words $w_s \cdots w_1 \in G_1 *_{G_3} G_2$ with $w_i \in G_1 \setminus G_3$ for even i and $w_i \in G_2 \setminus G_3$ for odd i.

The elements of $G_1 *_{G_3} G_2$ can be written in the form $w_s \cdots w_1$ with $w_1 \in G_1$, further $w_i \in G_1 \setminus G_3$ for odd $i > 1$ and $w_i \in G_2 \setminus G_3$ for even i. Suppose that a word $w_s \cdots w_1$ (as above) maps to $1 \in \mathrm{PGL}_2(K)$. Then $w_s \cdots w_1T = T$. What we have shown above implies that $w_1 \in G_3$. From $w_s \cdots w_2T = T$ one concludes that $s = 1$ and $w_1 = 1$. Thus the natural homomorphism $G_1 *_{G_3} G_2 \to \mathrm{PGL}_2(K)$ is injective.

The group $G_1 *_{G_3} G_2$ has a normal subgroup of finite index N which is a finitely generated free group. The group N is a Schottky group if every $\gamma \in N$, with $\gamma \neq 1$ is hyperbolic. In that case $G_1 *_{G_3} G_2$ is a discontinuous group. Take an element $\gamma \in N$, $\gamma \neq 1$. One can represent γ by some word $w_s \cdots w_1$ as above. After replacing γ by a conjugate we may suppose that this word is *cyclically reduced* which means that each w_i lies in $G_1 \cup G_2$ and not in G_3. Moreover consecutive w_i's are not in the same G_j and w_s, w_1 are not in the same G_j. For every $n \geqslant 1$ the element γ^n has length ns and the distance of T to γ^nT is $\geqslant nsc$. Suppose that γ is not hyperbolic, then there is a fixed lattice class S for γ. Let d be the distance of S to T. Then the distance of γ^nT to S is also d and therefore the distance between T and γ^nT is bounded by $2d$. This contradiction shows that γ is hyperbolic.

(2) One has to show that the canonical homomorphism $\Gamma^1 *_{\tau G_e} \Gamma^2 \to \mathrm{PGL}_2(K)$ is injective and that its image is a discontinuous group. The proof of (1) above remains valid if one replaces G_1, G_2, G_3, T by $\Gamma^1, \Gamma^2, \tau(G_e), V$. $\square$

In [5], a list is given of the amalgams $\Gamma = G_1 *_{G_3} G_2$ which can be realized as discontinuous subgroup of $\mathrm{PGL}_2(K)$ with $p_K = 0$. Our list 3.5 is longer, probably because there seems to be the additional condition in [5], that the quotient tree $\mathcal{T}/\Gamma$ (see 2.3) has two vertices and one edge.

Theorem 3.5. *All realizable amalgams $G_1 *_{G_3} G_2$ for $p_K = 0$ and $p_k > 0$. We will assume that $1 \neq G_3 \neq G_1, G_2$. Further $G_1 *_{G_3} G_2$ and $G_2 *_{G_3} G_1$ are considered as the same amalgam.*

(1) *The only cases with cyclic G_1 are*

$$C_{3m} *_{C_3} A_4,\ p_k = 3 \quad \textit{and} \quad C_{2m} *_{C_2} D_n,\ p_k = 2,\ n \textit{ odd}.$$

In the sequel we suppose that $G_1, G_2 \in \{A_5, S_4, A_4, D_n\}$.

(2) *$G_3 = C_m$ and C_m is a maximal cyclic subgroup of both G_1 and G_2.*

(3) *$D_{2m} *_{C_2} D_n$, $p_k = 2$, n odd and $D_{3m} *_{C_3} A_4$, $p_k = 3$ for $m > 1$. In both cases G_3 is not maximal cyclic in G_1.*

(4) *$p_k = 5$ and $A_5 *_{D_5} \{A_5, D_{5m}\}$ with $m > 1$.*

(5) *$p_k = 3$ and $\{A_5, S_4\} *_{D_3} \{A_5, S_4, D_{3m}\}$ with $m > 1$.*

(6) *$p_k = 2$ and $A_5 *_{A_4} \{A_5, S_4\}$, $S_4 *_{D_4} \{S_4, D_{4m}\}$, $D_{2n} *_{D_2} \{A_5, A_4, D_{2m}\}$ with $m, n > 1$ and odd n. Finally, $D_{2n} *_{D_2} S_4$ with odd n and such that the image of D_2 in S_4 is not contained in A_4.*

For a finite subgroup $G \subset \mathrm{PGL}_2(K)$ we write $\mathrm{br}(G)$ for the number of branch points of $\mathbf{P}^1_K \to \mathbf{P}^1_K/G$. For any realizable $\Gamma = G_1 *_{G_3} G_2$ is realizable the formula $\mathrm{br}(\Gamma) = \mathrm{br}(G_1) + \mathrm{br}(G_2) - \mathrm{br}(G_3)$ holds.

Proof. Suppose $p_K = 0$ and let G be a finite, non-cyclic subgroup of $\mathrm{PGL}_2(K)$. The reduction of $\mathbf{P}^1(K)$ with respect to the set $\mathcal{F}$ of its ramification points defines a subtree $\mathrm{Tree}(G)$ of the tree of $\mathbf{P}^1(K)$, i.e., the tree of all lattice classes in K^2. One can reconstruct most of $\mathrm{Tree}(G)$ from its quotient $\mathrm{Tree}(G)/G$ and the additional data of stabilizers and images of the points of $\mathcal{F}$. For a cyclic subgroup H of G one can determine where the axis of H lies with respect to $\mathrm{Tree}(G)$. For a non-cyclic subgroup H of G one can also determine the position of $\mathrm{Tree}(H)$ with respect to $\mathrm{Tree}(G)$.

The general method for obtaining a realizable $G_1 *_{G_3} G_2$ is to embed the two trees $\mathrm{Tree}(G_i)$, $i = 1, 2$, into the tree of $\mathbf{P}^1(K)$ in a way compatible with the common subgroup G_3 and such that one can apply Herrlich's criterion. In case G_3 is cyclic, this amounts to determining the G_3-axis for both $\mathrm{Tree}(G_i)$, $i = 1, 2$, and placing the invariant lattices classes $[M_i]$, $i = 1, 2$, for G_i, $i = 1, 2$, with respect to this G_3-axis such that Herrlich's criterion can be applied. For non-cyclic G_3 the situation is similar, but more complicated.

(1) Let $G_1 = C_{m\ell}$, $G_3 = C_\ell$ with $m, \ell > 1$. According to Theorem 3.3, G_1 must have less invariant lattice classes than G_3. Observation 3.4 yields that $\ell = p_k^s$. The G_3-axis is also the G_1-axis and therefore this axis can, in $\mathrm{Tree}(G_2)$, only intersect vertices with stabilizer G_3. This prevents G_2 from being cyclic. From the pictures of Section 2.3 one concludes that only $C_{3\ell} *_{C_3} A_4$ with $p_k = 3$ and $C_{2\ell} *_{C_2} A_4$ with $p_k = 2$ are candidates.

Let us consider the first case in more detail. The tree $\mathrm{Tree}(A_4)$ has a central point $[M_2]$ with stabilizer A_4. Connected to this there are 4 vertices with stabilizers the 4 subgroups of order 3. One fixes one of those vertices, say P, and calls its stabilizer C_3. From the position of the images of $\mathcal{F}$ one can see that the C_3-axis in $\mathrm{Tree}(A_4)$ only intersects in the vertex P.

The lattice class $[M_1]$ is put anywhere on the C_3-axis. We have now to find the lattice class T of Theorem 3.3 on the segment $[P, [M_2]]$. A small calculation shows that the distance between P and $[M_2]$ is $-\log|\zeta_3 - 1|$. In particular, the vertex $[M_2]$ is not stable under any $\alpha \in G_1 \setminus G_3$. Hence T exists and $C_{3\ell} *_{C_3} A_4$, $p_k = 3$ is realizable. The same proof works for the other candidate.

(2) Let C_m be a maximal cyclic subgroup of G_1. Then the points on the C_m-axis in Tree(G_1) which have a large enough distance to $[M_1]$, a chosen G_1-invariant lattice class, are only stabilized by C_m. The same holds if C_m is a maximal cyclic subgroup of G_2. If one places $[M_1]$ and the G_2-invariant lattice $[M_2]$ at great enough distance then there is a T on the C_m-axis, between $[M_1]$ and $[M_2]$, which is stabilized only by C_m.

(3) We suppose that $G_3 = C_\ell$ is cyclic but not maximally in G_1. Let $C_{m\ell}$ be the maximal cyclic subgroup of G_1 containing G_3. If $G_1 *_{G_3} G_2$ is realizable then also $C_{m\ell} *_{C_\ell} G_2$. From (1) one concludes that the only candidates are $D_{2m} *_{C_2} D_n$, $p_k = 2$, n odd, and $D_{3m} *_{C_3} A_4$. The proof that these two groups are realizable is similar to the proof in (1).

(4) Now we consider non-cyclic G_3's. Since G_3 stabilizes at least two lattice classes one concludes that $G_3 = D_{p_k^s}$ or A_4 with $p_k = 2$. In the latter case the only candidates are $G_1 *_{A_4} G_2$ with $G_1, G_2 \in \{A_5, S_4\}$. Using the pictures of Section 2.3 one concludes that the tree Tree(A_4) fits in two ways into Tree(S_4). The same holds for the embedding of Tree(A_4) and Tree(A_5) if one fixes A_4 as subgroup of A_5. Using this and Theorem 3.3, one obtains that $A_5 *_{A_4} \{A_5, S_4\}$, $p_k = 2$ are the only realizable amalgames with $G_3 = A_4$.

Next we suppose that $G_3 = D_{p_k^s}$ with $p_k > 2$. If both G_1 and G_2 are dihedral groups, say with fixed lattice classes $[M_1]$, $[M_2]$. The axis for an element C of order 2 in G_3 is the horizontal line of the corresponding picture of Section 2.3. The same holds for C considered as element of G_1 and G_2. In glueing Tree(G_1) over Tree(G_2), the invariant lattice classes for G_1 and G_2 are on the same position. This contradiction yields that we may suppose that G_1 is not a dihedral group and thus $G_3 = D_{p_k}$. For $p_k = 3, 5$ it can seen that all candidates are realizable.

In the last part of the proof $p_k = 2$. Consider first $G_3 = D_2$.

The essential part E of the tree Tree(D_2) is the set of vertices and edges which are invariant under D_2. The vertices of E are denoted by v_0, v_1, v_2, v_3 and the edges of E are $[v_0, v_i]$, $i = 1, 2, 3$. The automorphism group S_3 of D_2 acts faithfully on v_1, v_2, v_3. For $G \in \{A_5, S_4, A_4, D_{2m}\}$ and a given embedding $D_2 \subset G$ there is an embedding of Tree(D_2) in Tree(G). For each of the v_i we write $G(v_i)$ for the stabilizer of v_i in the group G. We note that there is still the freedom of permuting v_1, v_2, v_3. Consider two embeddings $D_2 \subset G_1, G_2$. This induces groups $G_1(v_i)$ and $G_2(v_i)$ for each i. If $G_1 *_{G_3} G_2$ is realizable then necessarily for each i at most one of the groups $G_j(v_i)$ is different from D_2. To show that this condition is also sufficient one considers an edge, say $[v_0, v_1]$, with $G_2(v_0) = D_2$ and $G_1(v_1) = D_2$. There is a point $T_1 \in [v_0, v_1]$ such that $\alpha \in G_1 \setminus D_2$ does not stabilize any point $P \neq T_1$ with $P \in [T_1, v_1]$. Similarly, there is a point T_2 such that $\beta \in G_2 \setminus D_2$ does not stabilize any point $P \neq T_2$ with $P \in [v_0, T_2]$. Now one has to verify that $[v_0, T_2] \cap [T_1, v_1]$ is a non-trivial segment. It happens that every case where the necessary condition is satisfied, this intersection is non-trivial. The details can easily be deduced from the information on the groups G. From the following table one can read off all realizable $G_1 *_{D_2} G_2$. A $*$ indicates that the corresponding group $G(v_i)$ is not equal to D_2. The first S_4 means that $D_2 \subset S_4$ is not contained in A_4 and the second S_4 indicates the opposite situation.

E	D_{2n}, odd n	D_{2m}, even m	S_4	S_4	A_5	A_4
v_0		*	*	*	*	*
v_1	*	*	*	*	*	
v_2			*	*	*	
v_3				*		

Next we consider $G_3 = D_4$. The essential part E of $\mathrm{Tree}(D_4)$ will be the part stabilized by D_4. This E has the same description as before. However one of the end vertices, say v_1, is "marked" by the position of the ramification points of order 4. An automorphism of D_4 permutes the other points v_2, v_3. For an embedding $D_4 \subset G$ one writes, as above, $G(v_i)$ for the stabilizer in G of the vertex v_i. For $G = D_{4m}$ one finds, due to this marking, that $D_{4m}(v_1) = D_{4m}$. Moreover $D_{4m}(v_0) \neq D_4$ if and only if m is even. Further $D_{4m}(v_2) = D_{4m}(v_3) = D_4$. In particular, for a realizable $G_1 *_{D_4} G_2$, one of the G_i is not a dihedral group. With the previous notation one has $S_4(v_i) \neq D_4$ only if $i = 2$. This shows that (omitting a small verification concerning the length of certain segments) the candidates $S_4 *_{D_4} \{S_4, D_{4m}\}$ are indeed realizable.

Finally, $G_3 = D_{2^s}$ with $s \geqslant 3$ is seen to be impossible by methods similar to the case of D_4.

The formula $\mathrm{br}(\Gamma) = \mathrm{br}(G_1) + \mathrm{br}(G_2) - \mathrm{br}(G_3)$ in situation (2) of 3.5, is a special case of the main result of Section 5. We *sketch* a method which proves the formula in all cases of 3.5. A realization embeds Γ as discontinuous subgroup of $\mathrm{PGL}_2(K)$. Let $\mathcal{F}_i$, $i = 1, 2$, denote the ramification points of G_i. Let $\mathcal{F}$ be the union of all Γ-orbits of $\mathcal{F}_1 \cup \mathcal{F}_2$. One considers the tree $\mathcal{T}_{\mathcal{F}}$ and the corresponding admissible affinoid covering $\{X_v, X_e \mid \text{all } v, e\}$ of Ω. A careful, case by case, analysis is needed to locate the affinoids of this covering which contain the points $\mathcal{F} \cap \Omega$. As an example we consider $p_k = 2$ and $G_1 *_{G_3} G_2$ with $G_1, G_2 \cong A_5$ and $G_3 \cong A_4$. From the given construction of the realization one can read off the following data. The tree $\mathcal{T}_{\mathcal{F}}$ can be seen to have four kinds of vertices $[M]$, namely with stabilizers $\Gamma_{[M]}$ conjugated to: (i) G_1, (ii) G_2, (iii) G_3, (iv) the subgroup of G_3, isomorphic to D_2.

$X_{[M]} \cap \mathcal{F}$ consists of two ramification points of order 5 in the cases (i) and (ii); is empty for case (iii) and consists of two ramification points of order 2 for case (iv). For an edge e, the set $X_e \cap \mathcal{F}$ is empty. We conclude that $\mathrm{br}(\Gamma) = 3$. □

Corollary 3.6. *Suppose that $p_K = 0$. Let $\Gamma \subset \mathrm{PGL}_2(K)$ be a finitely generated, discontinuous, infinite, indecomposable group such that $\Omega/\Gamma \cong \mathbf{P}^1(K)$. We exclude the following cases for the group Γ:*

(a) *$p_k = 2$ and Γ contains S_4 or A_5 or D_n with $n \neq 2^s$.*
(b) *$p_k = 3$ and Γ contains A_4.*
(c) *$p_k = 5$ and Γ contains A_5.*

Then every maximal finite subgroup of Γ is non-cyclic and has a unique separating lattice class. Every non-trivial intersection H of distinct maximal finite subgroups G_1, G_2 is a *maximal cyclic subgroup of Γ and moreover the canonical homomorphism $G_1 *_H G_2 \to \Gamma$ is injective.*

Proof. Let G be a maximal finite subgroup and let $[M] \in \mathcal{T}$ be stabilized by G. There is a finite subtree T of $\mathcal{T}$, containing $[M]$, which is mapped bijectively to $\mathcal{T}/\Gamma$. This implies that Γ contains a non-trivial amalgam $G_1 *_{G_3} G_2$ with $G = G_1$. By 3.5, G is not cyclic and moreover G has a unique separating lattice class.

Consider two maximal finite subgroups $G_1 \neq G_2$ with $H = G_1 \cap G_2 \neq 1$. Let $[M_1]$, $[M_2]$ denote the separating lattice classes for G_1 and G_2. First suppose that H contains an element h of order not divisible by p_k. Let h_0, h_1 denote the two fixed points of h. They determine the axis L of h in $\mathcal{BT}$. The group Γ_L consisting of the elements $\gamma \in \Gamma$ which have each point of L as fixed point is equal to $\{\gamma \in \Gamma \mid \gamma h_0 = h_0,\ \gamma h_1 = h_1\}$, is a discontinuous group, and has a maximal finite subgroup F. This group is a maximal cyclic subgroup of Γ. Since $[M_1]$ and $[M_2]$ are on L one has $H \supset F$. In case $H = F$ we are done. If $H \neq F$, then $H \cong D_{p_k^a}$ for some $a \geqslant 1$. Moreover $G_1 \cong D_n$ and $G_2 \cong D_m$ with $p_k^a \mid n$ and $p_k^a \mid m$. Further $[M_1]$ and $[M_2]$ are also separating lattices for the subgroup H of G_1 and of G_2. This contradicts the uniqueness of the separating lattice for H.

Suppose now that every element of H has order a power of p_k. Then H is cyclic. Further G_1 and G_2 are dihedral groups. From the pictures in Section 2.5 it follows that the H-axis L passes through the separating lattices $[M_1]$ and $[M_2]$. Further L is also pointwise invariant under the normal subgroups of index 2 (note that $p_k \neq 2$) of G_1 and G_2. We conclude that H is maximal cyclic in both G_1 and G_2.

Finally, we want to show that the canonical homomorphism $G_1 *_H G_2 \to \mathrm{PGL}_2(K)$ is injective. Let again $[M_1]$, $[M_2]$ denote the separating lattices for G_1 and G_2. Let $[M_3] \neq [M_1]$ be the vertex of $\mathcal{T}$ on the segment $[[M_1], [M_2]]$ in $\mathcal{BT}$, closest to $[M_1]$, such that the stabilizer $\Gamma_{[M_3]}$ of $[M_3]$ is a maximal finite subgroup of Γ and $[M_3]$ is its separating lattice. For any $[M] \neq [M_1]$, $[M_3]$ on the segment $[[M_1], [M_3]]$ the stabilizer $\Gamma_{[M]}$ is not a maximal finite subgroup of Γ or it is a maximal finite subgroup but $[M]$ is not separating for $\Gamma_{[M]}$. Clearly $\Gamma_{[M]} \supset H$. Suppose that $[M] \neq [M_1], [M_3]$, $[M] \in [[M_1], [M_3]]$ has the property that $\Gamma_{[M]}$ is not contained in either G_1 or G_3. The group $\Gamma_{[M]}$ is contained in (or equal to) a maximal finite subgroup G_4 with separating lattice $[M_4] \neq [M]$. The intersection $G_1 \cap G_4$ is maximal cyclic, contains H and is therefore equal to H. As seen above, $[M_1]$ and $[M_4]$ lie on the axis of the cyclic group H. Similarly $[M_3]$ and $[M_4]$ lie on the axis of H. However the vertices $[M_1]$, $[M_3]$, $[M_4]$ do not lie on a line of $\mathcal{BT}$. We conclude that $\Gamma_{[M]}$ lies in either G_1 or G_3 for every $[M] \in [[M_1], [M_3]]$.

For every $g \in G_1 \setminus H$ there is a lattice $[M(g)] \in [[M_1], [M_3]]$ such that g stabilizes all $[M] \in [[M_1], [M(g)]]$ and does not stabilize any other $[M]$ on $[[M_1], [M_3]]$. Let $[[M_1], [M_1^*]]$ denote the union of the $[[M_1], [M(g)]]$ for all $g \in G_1 \setminus H$. Then $\Gamma_{[M]} \subset G_1$ if and only if $[M] \in [[M_1], [M_1^*]]$ or $\Gamma_{[M]} = H$. Similarly, there is an $[M_3^*] \in [[M_1], [M_3]]$ such that $\Gamma_{[M]} \subset G_3$ if and only if $[M] \subset [[M_3^*], [M_3]]$ or $\Gamma_{[M]} = H$. Suppose that there is no $[M]$ with $\Gamma_{[M]} = H$, then $[[M_1], [M_3]]$ is the union of the segments $[[M_1], [M_1^*]]$, $[[M_3^*], [M_3]]$ and thus the intersection $[[M_1], [M_1^*]] \cap [[M_3^*], [M_3]]$ is not empty. A lattice class $[M]$ in this intersection satisfies $\Gamma_{[M]} \subset G_1 \cap G_3 = H$. This contradiction shows that $\Gamma_{[M]} = H$ holds for some $[M] \in [[M_1], [M_2]]$. According to the first part of Theorem 3.3 the homomorphism $G_1 *_H G_2 \to \mathrm{PGL}_2(K)$ is injective. $\square$

In the sequel of this section we suppose that $p_K = p_k = p > 0$ and our aim is to compute the list of all $G_1 *_{G_3} G_2$ which can be realized as discontinuous subgroup of $\mathrm{PGL}_2(K)$.

Proposition 3.7. *Suppose that $p_K = p > 0$. Let $\Gamma \subset \mathrm{PGL}_2(K)$ be a finitely generated, discontinuous, infinite indecomposable group such that $\Omega/\Gamma \cong \mathbf{P}^1(K)$. Then every maximal finite subgroup of Γ is non-cyclic.*

Proof. We may restrict our attention to a realizable $\Gamma = G_1 *_{G_3} G_2$ and prove that G_1 cannot be cyclic. We identify G_i for $i = 1, 2, 3$ with its image $H_i \subset \mathrm{PGL}_2(K)$. Suppose that G_1 is cyclic. Let $[M_1] \neq [M_2]$ denote G_1 and G_2 stable classes of lattices. We make the following observations:

(a) Let $g \in \mathrm{PGL}_2(K)$ have order l, $1 < l < \infty$, then $l = p$ or $p \nmid l$.
(b) Let $g \in \mathrm{PGL}_2(K)$ have order $l \geqslant 2$, not divisible by p. Let e_1, e_2 be a basis of K^2 consisting of eigenvectors. The collection of all g-stable lattice classes is $\{[K^0 e_1 + K^0 \lambda e_2] \mid \lambda \in K^*\}$. This is an "infinite line" in the tree of all classes of lattices.

The first observation prevents G_3 of having order p. The second observation implies that G_1 and G_3 have the same infinite line of stable classes of lattices. This line passes through $[M_1]$ and $[M_2]$, which contradicts Herrlich's criterion. □

Observation 3.8 (*p-Sylov subgroups*). Let G be a finite group which has an embedding in $\mathrm{PGL}_2(K)$ and let $C_m \subset G$ be a maximal cyclic subgroup with $m > 1$ and m not divisible by p. We fix an embedding $\phi : C_m \to \mathrm{PGL}_2(K)$, say with image $\{(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}) \mid a^m = 1\}$. The group $\phi(C_m)$ normalizes two Borel groups $B_+ = \{(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix})\}$, $B_- = \{(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix})\}$ and their unipotent subgroups $U_+ = \{(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix})\}$, $U_- = \{(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix})\}$. Let $\psi : G \to \mathrm{PGL}_2(K)$ be an embedding with extends ϕ. Define $U_\pm(G) = \{g \in G \mid \psi(g) \in U_\pm\}$. A group $U_\pm(G)$ is either trivial (i.e., $= \{1\}$) or a p-Sylov subgroup of G normalized by C_m. For $m > 2$ the groups $U_\pm(G)$ do not depend on ψ. For $m = 2$ one can change ψ into $\tilde{\psi}$ given by $\tilde{\psi}(g) = \psi(g^{-1})^*$, where $*$ means the transposed w.r.t. a basis of eigenvectors for $\phi(C_m)$. Thus for $m = 2$ one cannot distinguish between the two groups $U_\pm(G)$.

We give now a list of all possible pairs $C_m \subset G$ and $U_\pm(G)$:

(1) $B(n, m)$ with $m > 1$. Precisely one non-trivial $U_\pm(G)$.
(2) $\mathrm{PGL}_2(\mathbf{F}_q)$ with $q > 2$, $m = q - 1$. Both $U_\pm(G)$ are non-trivial.
(3) $\mathrm{PGL}_2(\mathbf{F}_q)$ and $m = q + 1$. Both $U_\pm(G)$ are trivial.
(4) $\mathrm{PSL}_2(\mathbf{F}_q)$ with $p \neq 2$ and $m = (q - 1)/2 > 1$. Both $U_\pm(G)$ are non-trivial.
(5) $\mathrm{PSL}_2(\mathbf{F}_q)$, $p \neq 2$ and $m = (q + 1)/2$. Both $U_\pm(G)$ are trivial.
(6) A_5, $p = 3$ and $m = 5$. Both $U_\pm(G)$ are trivial.
(7) A_5, $p = 3$ and $m = 2$. Both $U_\pm(G)$ are non-trivial.
(8) D_ℓ, $p = 2$, $m = \ell$ odd. Both $U_\pm(G)$ are trivial.
(9) $G \in \{D_n, A_4, S_4, A_5\}$, $p \nmid |G|$, C_m maximal cyclic subgroup. Both $U_\pm(G)$ are trivial.

This observation and the list will be used in the formulation of the next propositions.

Proposition 3.9. *Suppose $p_K = p > 0$. Consider an amalgam $\Gamma = G_1 *_{G_3} G_2$ with G_1, G_2 isomorphic to subgroups of $\mathrm{PGL}_2(K)$, which are not of Borel type. The only cases where Γ can be realized are*

(1) *G_3 is a maximal cyclic subgroup of both G_1 and G_2 of order $m > 1$ prime to p. Moreover all the groups $U_\pm(G_1)$, $U_\pm(G_2)$ are trivial.*
(2) *In addition for $p = 3$, the group $\mathrm{PSL}_2(\mathbf{F}_3) *_{C_3} \mathrm{PSL}_2(\mathbf{F}_3)$.*
(3) *In addition for $p = 2$, the groups $D_\ell *_{C_2} D_m$ with odd ℓ, m.*

Proof. Suppose that $\Gamma = G_1 *_{G_3} G_2$ is realizable as a discontinuous group. For convenience we identify G_i, $i = 1, 2, 3$, with their images in $\mathrm{PGL}_2(K)$. Let $[M_i]$, $i = 1, 2$, be the lattice classes invariant under G_i. Then $[M_1] \neq [M_2]$ are both stable under G_3 and thus G_3 lies in a Borel group $B \subset \mathrm{PGL}_2(K)$.

(1) Suppose that p does not divide the order of G_3. Then G_3 is cyclic of order $m > 1$ with $p \nmid m$. By Theorem 3.3, G_3 is a maximal cyclic subgroup of both G_1 and G_2 (and thus of Γ). Suppose that, say, both $U_+(G_1)$, $U_+(G_2)$ are non-trivial. These groups lie in a common Borel subgroup of $\mathrm{PGL}_2(K)$ and generate a finite subgroup of Γ. This subgroup must be conjugated to a subgroup of either G_1 or G_2. Since this is clearly not the case, this possibility is excluded.

Suppose that both $U_\pm(G_1)$ are non-trivial. The two invariant lattices have the form $M_1 = K^0 e_1 + K^0 e_2$ and $M_2 = K^0 e_1 + K^0 \pi e_2$, where e_1, e_2 is a basis of K^2 consisting of eigenvectors for the group C_m. After possibly interchanging e_1 and e_2 one finds that $U_+(G_1)$ stabilizes M_2. Therefore the group $G \subset \Gamma$ generated by $U_+(G_1)$ and G_2 stabilizes M_2. By assumption Γ is discontinuous and so G is a finite group. However no conjugate of G is contained in either G_1 or G_2. This contradicts that $\Gamma = G_1 *_{G_3} G_2$. In view of the list in Observation 3.8 we conclude that the groups $U_\pm(G_1)$, $U_\pm(G_2)$ are trivial.

In order to show that the conditions are sufficient, we apply Herrlich's criterion. One fixes an embedding of $G_3 = C_m$ in $\mathrm{PGL}_2(K)$. Let e_1, e_2 denote two independent eigenvectors of C_m. One chooses the lattices $M_1 = K^0 e_1 + K^0 e_2$ and $M_2 = K^0 \pi e_1 + K^0 e_2$ and embeddings of G_1, G_2 such that for $i = 1, 2$ the lattice M_i is invariant under G_i and with $0 < |\pi| < 1$. The conditions of Theorem 3.3 are satisfied as is easily seen from Proposition 2.1 and direct computation.

(2) Suppose that p divides the order of G_3. Suppose that the groups G_i, $i = 1, 2$, are isomorphic to $\mathrm{PHL}_2(\mathbf{F}_{q_i})$ where H denotes G or S and q_1, q_2 are powers of p. Choose a basis e_1, e_2 of K^2 such that G_1 is the subgroup $\mathrm{PHL}(\mathbf{F}_{q_1} e_1 + \mathbf{F}_{q_1} e_2)$ of $\mathrm{PGL}_2(K)$ and such that G_3 contains the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ w.r.t. this basis. The group G_2 can then be identified with $\mathrm{PHL}(\mathbf{F}_{q_2} e_1 + \mathbf{F}_{q_2} e_3)$ for some $e_3 = a e_1 + b e_2$. Using that $G_3 \subset G_2$ one finds that $b \in \mathbf{F}_{q_2}^*$. We may then suppose that $b = 1$. Moreover $a \neq 0$ and $|a| > 1$. Indeed, otherwise the lattice $K^0 e_1 + K^0 e_2$ is stabilized by both G_1 and G_2 and thus the group generated by G_1 and G_2 cannot be an infinite discontinuous group. The intersection of G_1 and G_2 is easily seen to be $\{\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right) \mid y \in \mathbf{F}_{q_1} \cap \mathbf{F}_{q_2}\}$. Let B be the Borel subgroup of $\mathrm{PGL}_2(K)$ which contains the element $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \in G_3$. Then $B \cap G_1$ and $B \cap G_2$ lie in B and generate a finite subgroup T of Γ. A conjugate of T should lie in either G_1 or G_2. Suppose that a conjugate of T lies in G_2. Then $T \subset G_2$. Thus $B \cap G_1 \subset B \cap G_2 = T$ and $G_3 \supset B \cap G_1$. Since the elements of G_3 have order 1 or p, the same must hold for $B \cap G_1$. This is only possible for $p = 2$ and $G_1 = \mathrm{PSL}_2(\mathbf{F}_2) \cong D_3$ or $p = 3$ and $G_1 = \mathrm{PSL}_2(\mathbf{F}_3) \cong A_4$.

The element $\binom{1\ 1}{0\ 1}$ stabilizes both lattices $M_1 = K^0e_1 + K^0e_2$ and $M_2 = K^0e_1 + K^0(ae_1 + e_2)$. Then also $B \cap G_2$ stabilizes M_1. The subgroup G of Γ generated by G_1 and $B \cap G_2$ is finite since Γ is discontinuous and G stabilizes M_1. Therefore G is finite and a conjugate of G must lie in either G_1 or G_2. This can only be the case when $G = G_1$ and we conclude that there are only two possibilities left, namely $\mathrm{PSL}_2(\mathbf{F}_2) *_{C_2} \mathrm{PSL}_2(\mathbf{F}_2)$ and $\mathrm{PSL}_2(\mathbf{F}_3) *_{C_3} \mathrm{PSL}(\mathbf{F}_3)$.

Take an element $\pi \in K$ with $0 < |\pi| < 1$ and consider the groups

$$G_1 = \mathrm{PSL}(\mathbf{F}_2e_1 + \mathbf{F}_2e_2), \quad G_2 = \mathrm{PSL}\bigl(\mathbf{F}_2\pi e_1 + \mathbf{F}_2(e_1 + \pi e_2)\bigr)$$

and the lattices

$$M_1 = K^0e_1 + K^0e_2 = K^0(e_1 + \pi e_2) \quad \text{and}$$
$$M_2 = K^0\pi e_1 + K^0(e_1 + \pi e_2) = K^0(e_1 + \pi e_2) + K^0\pi^2 e_2.$$

For $i = 1, 2$ the lattice M_i is stable under G_i. Any lattice class $[M_3] \neq [M_1], [M_2]$ lying in the segment $[[M_1], [M_2]]$ has the form $M_3 = K^0(e_1 + \pi e_2) + K^0\lambda e_2$ where $|\pi^2| < |\lambda| < 1$. One easily verifies that the only non-trivial element in $G_1 \cup G_2$ which stabilizes $[M_3]$ is represented by the matrix $\binom{1\ 1}{0\ 1}$ with respect to the basis e_1, e_2. Therefore the conditions of Theorem 3.3 are satisfied and we conclude that $\mathrm{PSL}_2(\mathbf{F}_2) *_{C_2} \mathrm{PSL}_2(\mathbf{F}_2)$ can be realized as a discontinuous subgroup of $\mathrm{PGL}_2(K)$. The same method works for the group $\mathrm{PSL}_2(\mathbf{F}_3) *_{C_3} \mathrm{PSL}(\mathbf{F}_3)$.

(3) For $p = 3$ a new possibility for G_1 or G_2 occurs, namely the group A_5. Arguments using Borel subgroups of $\mathrm{PGL}_2(K)$, as in (2) above, exclude this possibility.

(4) For $p = 2$ the new possibility for G_1 or G_2 is D_ℓ with odd ℓ. Arguments using Borel subgroups, as in (2), exclude all possibilities, except for $D_\ell *_{C_2} D_m$ with odd m. For these groups one can verify the conditions of Theorem 3.3. This exhaust all combinations. □

Proposition 3.10. *Suppose that $p_K = p > 0$, that G_1 has type $B(n,m)$ with $n > 0$ and $m \geqslant 1$. The list of the amalgams $\Gamma = G_1 *_{G_3} G_2$ which can be realized as discontinuous subgroups of* $\mathrm{PGL}_2(K)$ *is the following*:

(1) *$m > 1$, $G_3 = C_m$. Fix an embedding of C_m in* $\mathrm{PGL}_2(K)$ *and let $U_+(G_1)$ be the normal p-Sylov group of G_1. Then G_3 is a maximal cyclic subgroup of G_2 and $U_+(G_2)$ is trivial. For $m = 2$, this condition can also be formulated as*: *one of the groups $U_\pm(G_2)$ is trivial.*
(2) *$m \geqslant 1$ and $G_2 = \mathrm{PHL}_2(\mathbf{F}_q)$ with $H \in \{G, S\}$. Let B denote the Borel subgroup of* $\mathrm{PHL}_2(K)$. *Then $G_3 = B(\mathbf{F}_q)$ and $m = q - 1$ if $H = G$ and $m = (q-1)/2$ if $H = S$ and $p \neq 2$.*
(3) *In addition for $p = 3$, the groups $B(n, 2) *_{D_3} A_5$.*
(4) *In addition for $p = 2$, the groups $B(n, 1) *_{C_2} D_\ell$ with odd ℓ.*

Proof. (i) The proof of part (1) is similar to the one of part (2) of Theorem 3.5. Now we have to consider the case that p divides the order of G_3.

(ii) Suppose first that G_2 is also of type $B(\tilde{n},\tilde{m})$. Then G_1 and G_2 lie in the same Borel subgroup B of $\mathrm{PGL}_2(K)$ and generate therefore a finite group, which is not isomorphic to $G_1 *_{G_3} G_2$.

We conclude that G_2 is not of type $B(\tilde{n},\tilde{m})$. Since p divides the order of G_2, the remark following Proposition 2.1 gives the possibilities for G_2.

(iii) Suppose that G_2 has the form $\mathrm{PGL}(\mathbf{F}_q e_1 + \mathbf{F}_q e_2)$, where e_1, e_2 is a basis of K^2 over K. This basis is chosen such that some element of G_3 has the matrix $\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. The unique Borel subgroup $B \subset \mathrm{PGL}_2(K)$ which contains G_3 consists of all matrices with e_1 as eigenvector. Also $G_1 \subset B$ and the subgroup generated by G_1 and $B(\mathbf{F}_q)$ is finite. Since G_1 is a maximal finite subgroup of $G_1 *_{G_3} G_2$ one has that $G_1 \supset B(\mathbf{F}_q)$ and therefore $G_3 = B(\mathbf{F}_q)$. In particular, m is a multiple of $q-1$.

Write $m = d(q-1)$ with $d \geqslant 1$. Let $[M_1] \neq [M_2]$ denote the lattice classes stabilized by G_1 and G_2. Take an element $a \in G_1$ with order $d(q-1)$. Then $a^d \in G_3$ stabilizes both $[M_1]$ and $[M_2]$. Since the order of a is not divisible by p, also a stabilizes $[M_2]$ and the group generated by a and G_2 stabilizes $[M_2]$. Since we have supposed that $G_1 *_{G_3} G_2$ is realizable, this implies that $a \in G_2$. We conclude that $m = q-1$. The same reasoning holds for $G_2 = \mathrm{PSL}_2(\mathbf{F}_q)$. We conclude that the amalgams in part (2) of the present proposition are the only candidates.

In order to show that the candidates pass the test of Theorem 3.3, we consider an example. The general case can be treated in the same way. For this example we take $G_2 = \mathrm{PGL}(\mathbf{F}_q e_1 + \mathbf{F}_q e_2)$ and we take for G_1 the group, given by matrices $\{\left(\begin{smallmatrix}a&b\\0&1\end{smallmatrix}\right) \mid a \in \mathbf{F}_q^*,\ b \in \mathbf{F}_q + \pi^{-1}\mathbf{F}_q\}$ with respect to the basis e_1, e_2. Here $0 < |\pi| < 1$. Consider the two lattices $M_1 = K^0 e_1 + K^0 \pi e_2$ and $M_2 = K^0 e_1 + K^0 e_2$, invariant under respectively G_1 and G_2. The verification of the conditions of Theorem 3.3 is immediate.

(iv) For $p = 3$, the new possibility is $B(n,m) *_{G_3} A_5$, where 3 divides the order of G_3. Suppose that this group is realizable. As in part (iii) of the proof one finds that $G_3 = D_3$ (i.e., the intersection of a Borel group with A_5) and $m = 2$. The verification that $B(n,2) *_{D_3} A_5$ satisfies the conditions of Theorem 3.3 is similar to the verification in (iii).

(v) For $p = 2$ we have to consider the possibility $B(n,m) *_{G_3} D_\ell$ with odd ℓ. Since $G_3 \subset D_\ell$, one must have $G_3 = C_2$. Let $[M_1] \neq [M_2]$ denote the lattice classes, stabilized by respectively G_1 and G_2. Let $A = \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$ belong to $G_3 = C_2$ and let ζ_m denote a primitive mth-root of unity. Then some $B = \left(\begin{smallmatrix}\zeta_m&x\\0&1\end{smallmatrix}\right)$ belongs to G_1 and $C = BAB^{-1} = \left(\begin{smallmatrix}1&\zeta_m\\0&1\end{smallmatrix}\right) \in G_1$. Clearly C also stabilizes $[M_2]$ and the group generated by C and G_2 is finite since we have supposed that $G_1 *_{G_3} G_2$ is realizable. Since G_2 is a maximal finite subgroup of $G_1 *_{G_3} G_2$ one has $m = 1$. We conclude that the only candidate is $B(n,1) *_{C_2} D_l$. As in (iii), one verifies the conditions of 3.3. □

Corollary 3.11. *The list of discontinuous groups $G_1 *_{G_3} G_2$ for $p_K = p > 0$.*

(1) $G_3 = C_m$, $m > 1$, $p \nmid m$, C_m *maximal cyclic subgroup of both* G_1, G_2 *and satisfying the condition on the groups* $U_\pm(G_1)$, $U_\pm(G_2)$ *of* 3.9 *and* 3.10. *Observation* 3.8 *provides all possibilities.*
(2) $B(n,q-1) *_{B(\mathbf{F}_q)} \mathrm{PGL}_2(\mathbf{F}_q)$.
(3) $B(n,(q-1)/2) *_{B(\mathbf{F}_q)} \mathrm{PSL}_2(\mathbf{F}_q)$ *for* $p \neq 2$.

(4) *For* $p = 3$ *additionally*: $\mathrm{PSL}_2(\mathbf{F}_3) *_{C_3} \mathrm{PSL}_2(\mathbf{F}_3)$ *and* $B(n,2) *_{D_3} A_5$.
(5) *For* $p = 2$ *additionally*: $D_\ell *_{C_2} D_m$, $D_\ell *_{C_2} B(n,1)$ *with odd* ℓ, m. (*For* $q = 2$, $B(n,1) *_{C_2} D_3$ *coincides with* $B(n, q-1) *_{B(\mathbf{F}_q)} \mathrm{PGL}_2(\mathbf{F}_q)$.)

The next two theorems extend the above results to more complicated finite indecomposable trees of groups.

Theorem 3.12. *Contracted finite, indecomposable trees for* $p_K = 0$. *Suppose* $p_K = 0$. *Let the finite tree of groups* (T, G) *satisfy the conditions* (a)–(f) *below. Then* (T, G) *can be realized in* $\mathcal{BT}$.

(a) $G_v \in \{D_n, A_4, S_4, A_5\}$ *for every vertex* v *of* T.
(b) *If* $p_k \neq 2$ *and* $p_k \mid \#G_v$, *then* G_v *is a dihedral group.*
(c) *If* $p_k = 2$, *then* $G_v \in \{A_4, D_{2^s}$ *with* $s \geqslant 1\}$.
(d) G_e *is a maximal cyclic subgroup of* G_v, *if* e *is an edge of* v.
(e) *If a maximal cyclic subgroup* H *of* G_v *is equal to some* G_e, *then there is at most one edge* $e' \neq e$ *with* $G_{e'} = H$. *Moreover, no conjugate* $H' \neq H$ *of* H *is equal to* $G_{e''}$ *for some edge* e'' *of the vertex* v.
(f) *If there are two edges* $e \neq e'$ *with vertex* v *and* $G_e = G_{e'} = H$, *then the normalizer of* H *in* G_v *is* H *itself.*

Proof. We will prove by induction on the number of edges of (T, G) that a realization τ exists. By Theorem 3.5 we may suppose that there are at least two edges. Let $e = \{v_1, v_2\}$ be an edge. Define the trees of groups (T^i, G^i), $i = 1, 2$, obtained by deleting e from (T, G) and such that v_i is a vertex of T^i. First we embed $H := G_e$ in $\mathrm{PGL}_2(K)$. Then one chooses two lattice classes $[M_1]$, $[M_2]$ on the axis $L \subset \mathcal{BT}$ of G_e having a large enough distance. By induction the (T^i, G^i), $i = 1, 2$, are realized in $\mathcal{BT}$ such that v_1, v_2 are mapped to $[M_1]$, $[M_2]$. Now we verify condition (2b) of 3.3.

Let $\Gamma^i \subset \mathrm{PGL}_2(K)$ be the realization of the amalgam of (T^i, G^i). The group $H^i := \{g \in \Gamma^i \mid gL = L\}$ has a subgroup of index 1 or 2 consisting of the elements of Γ^i commuting with H. By assumption Γ^i is the amalgam of (T^i, G^i). The elements of the amalgam of (T^i, G^i) can uniquely be represented by (suitably chosen) reduced words, as in the case of an amalgam of the form $G_1 *_{G_3} G_2$. Using this and the properties (d), (e) and (f), one shows that $\{\gamma \in \Gamma^i \mid \gamma$ commutes with $H\}$ is a finite group. Hence H^i is finite, too.

One considers the subtree $\mathcal{T}^i$ of $\mathcal{BT}$ generated by the Γ^i-orbits of all the embedded vertices of (T^i, G^i). We claim that $S^i := \mathcal{T}^i \cap L$ is a finite set. Suppose that S^i is infinite. The tree $\mathcal{T}^i$ is generated by the set of vertices $\{\gamma v \mid \gamma \in \Gamma^i, v$ a vertex of $T^i\}$. It follows that there are also infinitely many elements of this set lying on L. Hence there exists a vertex v of T^i and there are infinitely many $\gamma \in \Gamma^i$ such that $H \subset G_{\gamma v}$. Thus there are infinitely many $\gamma \in \Gamma^i$ with $\gamma H \gamma^{-1} \subset G_v$. The latter yields the contradiction that $\{\gamma \in \Gamma^i \mid \gamma$ commutes with $H\}$ is infinite. (In particular, the two fixed points of $H = G_e$ are not limit points for Γ^i.)

The distance between $[M_1]$, $[M_2]$ on L is taken large enough and thus there exists a lattice class V on $[[M_1], [M_2]]$, such that $S^1 \subset [[M_1], V)$, $S^2 \subset (V, [M_2]]$ and $g_i V = V$

with $g_i \in \Gamma^i$ implies $g_i \in H$. For any subtree S of $\mathcal{BT}$ there is an obvious projection $\mathrm{pr}_S : \mathcal{BT} \to S$. We write pr^1 for the projection on the subtree $\mathcal{T}^1$. Clearly, $\mathrm{pr}^1(V)$ lies in $S^1 \cap [[M_1], V)$.

For $g_1 \in \Gamma^1 \setminus H$ one has $\mathrm{pr}^1(g_1V) = g_1\,\mathrm{pr}^1(V)$ and the distance of V to $\mathcal{T}^1$ is the same as the distance of g_1V to $\mathcal{T}^1$. If V lies on $[g_1V, g_1\,\mathrm{pr}^1(V)]$, then $V = g_1V$, which is excluded by the choice of V. The path $[g_1V, g_1\,\mathrm{pr}^1(V)]$ followed by the path in $\mathcal{T}^1$ from $g_1\,\mathrm{pr}^1(V)$ to S^1 does not contain V. Thus $\mathrm{pr}_L(g_1V)$ lies on the left hand side of V. Similarly, for any $g_2 \in \Gamma_2 \setminus H$, the $\mathrm{pr}_L(g_2V)$ lies in on the right hand side of V. We conclude that $g_1V \neq V \neq g_2V$ and that $V \in [g_1V, g_2V]$. This is condition (2b) of Theorem 3.3. □

Remarks 3.13.

(1) It is an exercise to show that condition (f) is equivalent to: Let $x_1, x_2 \in \mathbf{P}^1(K)$ denote the two fixed points of the maximal cyclic subgroup H of G_v. Then the images of x_1, x_2 under the canonical map $\mathbf{P}^1(K) \to \mathbf{P}^1(K)/G_v \cong \mathbf{P}^1(K)$ are distinct.

(2) Consider for $n > 2$, the amalgam $D_n *_{C_n} D_n *_{C_n} D_n$. An embedding of this group in $\mathrm{PGL}_2(K)$ is easily seen to be conjugated to a group with generators τ, σ_i, $i = 1, 2, 3$, where $\tau z = \zeta z$ and ζ is a primitive nth root of unity, $\sigma_i z = a_i z^{-1}$ with “independent” $a_1, a_2, a_3 \in K^*$. This group is not discontinuous because it contains the elements $z \mapsto \frac{a_1}{a_2} z$ and $z \mapsto \frac{a_1}{a_3} z$. This example explains condition (f) of the theorem. A similar example shows that condition (e) is needed in the theorem.

(3) Suppose $p_k > 3$. Let (T, G) denote the tree of groups with vertices v_1, v_2, v_3, edges $e_1 = \{v_1, v_2\}$, $e_2 = \{v_2, v_3\}$ and groups $G_{v_1} = G_{v_2} = D_3$, $G_{v_3} = A_4$ and $G_{e_1} = G_{e_2} = C_3$ with the obvious embeddings in G_{v_1} and G_{v_2} and any embedding in G_{v_3}. Then (T, G) is not realizable. Let $(T, G)'$ denote the same tree of groups but with G_{v_2}, G_{v_3} interchanged. Then $(T, G)'$ is realizable. We note that the two trees of groups have the same amalgam!

Theorem 3.14. *Contracted finite, indecomposable trees for $p_K = p > 0$. Let $p_K = p > 0$. Suppose that the finite tree of groups (T, G) satisfies the conditions below, then (T, G) is realizable in $\mathcal{B}T$.*

If $p \geqslant 5$, then

(i) *Any vertex group G_v is isomorphic to a finite non-cyclic subgroup of* $\mathrm{PGL}_2(K)$. *In the sequel we will view the G_v's and G_e's as subgroups of* $\mathrm{PGL}_2(K)$ *and write $\phi_v : \mathbf{P}^1_K \to \mathbf{P}^1_K / G_v$ for the canonical morphism.*

(ii) *For any edge $e = \{v_1, v_2\}$ one has $1 \neq G_e \neq G_{v_1}, G_{v_2}$ and G_e is of Borel type. If $p \mid \#G_e$, then the group G_{v_i} is of Borel type for precisely one $i \in \{1, 2\}$.*

(iii) *Suppose that the vertex group G_v is not of Borel type, then*
   (a) *For any edge e of v the group G_e is a ramification group of ϕ_v.*
   (b) *Suppose that a ramification group $H \subset G_v$ of ϕ_v is equal to G_e. Then there is at most one edge $e' \neq e$ with $G_{e'} = H$. Moreover, no conjugate $H' \neq H$ of H is equal to some $G_{e''}$.*

(c) *If the edges $e \neq e'$, with vertex v, satisfy $H := G_e = G_{e'}$, then H has two fixed points $x_1 \neq x_2 \in \mathbf{P}^1(K)$ and $\phi_v(x_1) \neq \phi_v(x_2)$.*

(iv) *If the vertex group G_v is of Borel type and is not a p-group, then v is an extremal vertex.*

(v) *If the vertex group G_v is a p-group, then $G_e \cong C_p$ for every edge e containing v.*

If $p = 3$, then we admit also vertices v with $G_v = C_3$. Let $\{v, v_i\}_{i=1,\dots,m}$ denote the edges of v. We require that $m \geqslant 2$, that $G_{v_i} = \mathrm{PSL}_2(\mathbf{F}_3)$ and $G_{\{v,v_i\}} = C_3$ for all i. Moreover we exclude edges $e = \{v_1, v_2\}$ with groups $G_{v_1}, G_{v_2} \cong \mathrm{PSL}_2(\mathbf{F}_3)$ and $G_e \cong C_3$.

If $p = 2$, then we admit also vertices v with $G_v = C_2$. Let $\{v, v_i\}_{i=1,\dots,m}$ denote the edges of v. We require that $m \geqslant 2$, that $G_{v_i} = D_{\ell_i}$ with odd ℓ_i and that $G_{\{v,v_i\}} = C_2$ for all i. Moreover we exclude edges $e = \{v_1, v_2\}$ with groups $G_{v_1} \cong D_\ell$, $G_{v_2} \cong D_{\ell'}$, $G_e \cong C_2$ and odd ℓ, ℓ'.

Remarks 3.15 (*The special features for $p_K = 2, 3$*).

(1) For $p_K = 2$ and every $m \geqslant 2$ the tree of groups (T, G) with vertices $v, v_1, \dots, v_m$, edges $e_i = \{v_i, v\}$, $i = 1, \dots, m$, and groups $G_v = G_{e_i} = C_2$, $i = 1, \dots, m$, and $G_{v_i} = D_{\ell_i}$, $i = 1, \dots, m$, with odd ℓ_i's can be realized in $\mathcal{BT}$. One can prove this as follows: Let ℓ be the l.c.m. of $\ell_1, \dots, \ell_m$. Take $n > 1$ sufficiently large. The tree of groups with vertices v_1, v_2, edge $e = \{v_1, v_2\}$ and groups $G_{v_1} = D_\ell$, $G_e = C_2$, $G_{v_2} = C_2^n$ has a realization τ according to 3.10 part (4). In $\mathcal{BT}$ one considers the locally finite subtree T generated by the images of $\tau(v_1), \tau(v_2)$ under the action of $\Gamma = D_\ell *_{C_2} C_2^n$. The vertex $\tau(v_2)$ has $\#(C_2^n/C_2)$ edges. The stabilizer of each edge is the same group C_2. The stabilizer of each end point $\neq \tau(v_2)$ of an edge, is isomorphic to D_ℓ. We select now m of those edges $e_1, \dots, e_m$ and consider for each i a subgroup $G_i = D_{\ell_i}$ of D_ℓ which contains the fixed subgroup C_2 of C_2^n. This is an embedding of (T, G) in $\mathcal{BT}$. It is a realization, since the homomorphism of the amalgam of (T, G) to $D_\ell *_{C_2} C_2^n$ is injective and the latter group is discontinuous.
The same method can be used to realize the tree of groups (T, G) with vertices $v, v_1, \dots, v_m$, edges $e_i = \{v_i, v\}$, $i = 1, \dots, m$, and groups $G_v = C_2^n$ (any $n \geqslant 1$), $G_{e_i} = C_2$, $i = 1, \dots, m$, and $G_{v_i} = D_{\ell_i}$, $i = 1, \dots, m$, with odd ℓ_i's in $\mathcal{BT}$.

(2) For $p_K = 3$ and every $m \geqslant 2$ the tree of groups with vertices $v, v_1, \dots, v_m$, edges $e_i = \{v_i, v\}$, $i = 1, \dots, m$, and groups $G_v = C_3^n$ (any $n \geqslant 1$), $G_{e_i} = C_3$, $i = 1, \dots, m$, and $G_{v_i} = \mathrm{PSL}_2(\mathbf{F}_3)$, $i = 1, \dots, m$, can be realized in $\mathcal{BT}$.
Similar to (1) above, one proves this by means of the realizable amalgam $\mathrm{PSL}_2(\mathbf{F}_3) *_{C_3} C_3^{kn}$ with say $k > 1$ sufficiently large.

(3) The general idea for the formulation of Theorem 3.14 is that the tree of groups (T, G) can be realized and that no contraction of an edge in (T, G) is possible. We make a small exception for these rule, namely for technical reasons we allow that $m = 2$ in part (v) of Theorem 3.14. For the special cases $p_K = p = 2, 3$ one would like to contract the vertex with group C_p and all its edges. However, for $m > 2$ this introduces cycles and the new object is no longer a tree of groups.
According to 3.9 and 3.10, only for $p_K = 2, 3$ this special feature can occur.

Proof. The existence of a realization of (T, G) is proved by induction on the number of edges of T. By 3.11, we may suppose that T has at least two edges. For an edge e with $p \nmid \#G_e$, one can apply the method of the proof of 3.12.

Suppose that p divides $\#G_e$ for every edge e, that $G_e \neq G_v$ if e is an edge of v and that for no vertex v the group G_v is a p-group. Consider an edge $e = \{v_1, v_2\}$. By (ii), $G_{v_1} \cong B(n, m)$ with $m > 1$ and G_{v_2} is not of Borel type. By (iv), v_1 is an extremal edge. Let (T^1, G^1) be the tree of groups obtained by deleting v_1 and e. This tree is given a realization. It is not difficult to see that G_{v_1} can be embedded in $\mathrm{PGL}_2(K)$, such that condition (2b) of Theorem 3.3 is satisfied. In particular, for $p_K \geqslant 5$ the theorem is proved.

For $p_K = 2$ or 3, one first makes realizations of the subtrees of (T, G) which have the form described in Remarks 3.15, part (1) or (2). By induction and with 3.3, one can complete this to a realization of all of (T, G). $\square$

Definition 3.16. A *contracted finite, indecomposable tree of groups* is a tree of groups satisfying the conditions of Theorems 3.12 or 3.14 (depending on p_K and p_k).

4. The trees $\mathcal{T}$, $\mathcal{T}^c$, $\mathcal{T}^\dagger$ associated to Ω and Γ

As before, we consider an infinite, finitely generated discontinuous group $\Gamma \subset \mathrm{PGL}_2(K)$ such that Ω/Γ is isomorphic to $\mathbf{P}^1_K$. The aim of this section to find the structure of Γ. As we have seen, it suffices to consider an indecomposable group Γ such that its set of limit points has more than two elements. Let $\mathcal{T}$ denote the tree associated to Γ, defined in 2.3. The quotient $\mathcal{T}/\Gamma$ is a finite tree and we fix an embedding of $\mathcal{T}/\Gamma$ into $\mathcal{T}$. This makes $\mathcal{T}/\Gamma$ into a tree of groups. Since Γ is indecomposable every vertex and edge of $\mathcal{T}/\Gamma$ has a non-trivial stabilizer. In general, this tree of groups does not satisfy the properties stated in Theorems 3.12 or 3.14. Another tree $\mathcal{T}^c$, on which Γ acts, is constructed directly from the group Γ. Eventually, it will be shown that the tree of groups $\mathcal{T}^c/\Gamma$ has the properties of Theorems 3.12 or 3.14. In other words the structure of the above groups Γ has been established. In order to link $\mathcal{T}^c$ with a pure affinoid covering of Ω, we will have to consider still another tree $\mathcal{T}^\dagger$. There are *exceptional groups* Γ for which this construction does not work. The exceptional groups occur for $p_K = 0$ and $p_k = 2, 3, 5$. For these groups it seems rather difficult to find a structure theorem and a general formula for the number of branch points $\mathrm{br}(\Gamma)$ of Γ. In the paper [6] the exceptional groups with $\mathrm{br}(\Gamma) = 3$ are studied. We will introduce the notion of *ordinary* group Γ (which excludes the exceptional groups), carry out the constructions of $\mathcal{T}^c$ and $\mathcal{T}^\dagger$ and prove that the tree of groups $\mathcal{T}^c/\Gamma$ satisfies the properties of Theorems 3.12 or 3.14. *In the sequel of this section, Γ will denote* (*unless otherwise stated*) *a finitely generated, infinite, indecomposable, discontinuous subgroup of* $\mathrm{PGL}_2(K)$. *Moreover we will assume that the set of its limit points $\mathcal{L}$ is infinite.* For the omitted case, where Γ has two limit points, $\mathrm{br}(\Gamma)$ is known, see 2.3.

Definition 4.1. The group Γ will be called *ordinary* if every maximal finite subgroup, which is not of Borel type, has a separating lattice class. A few consequences of the property "ordinary" are

(i) No maximal finite subgroup of Γ is cyclic.

For $p_K = 0$ this follows from Theorem 3.5, since for $p_k = 3$ the group A_4 is excluded and for $p_k = 2$ the group D_n with odd n is excluded. For $p_K > 0$, this is proven in Proposition 3.7.

(ii) Let H be a maximal finite subgroup, which is not of Borel type, and H' is any maximal finite subgroup, not conjugated to H in Γ, with $H \cap H' \neq 1$. Then there exists a Borel group $B \subset \mathrm{PGL}_2(K)$ such that $H \cap H' = H \cap B$.

For $p_K = 0$ this follows from Theorem 3.5 since for $p_k = 2, 3$ or 5, the maximal finite subgroups for which (ii) does not hold are excluded. For $p_K > 0$ and H' not of Borel type, (ii) follows from 3.9. For $p_K > 0$ and H' of Borel type, statement (ii) follows from 3.10.

(iii) Let H be a maximal finite subgroup of Γ, which is not of Borel type. Then the ramification groups of the map $\varphi_H : \mathbf{P}^1_K \to \mathbf{P}^1_K / H$ are the subgroups $B \cap H \neq 1$ with $B \subset \mathrm{PGL}_2(K)$ a Borel group.

For $p_K = 0$ this follows from (i) above. For $p_K > 0$, this follows from (ii) above.

The proposition below shows that ordinary groups are very common indeed.

Proposition 4.2. *The group Γ is ordinary if and only if one of the following statements holds*:

(i) $p_K = p > 0$ *or* $p_K = 0$ *and* $p_k > 5$.
(ii) $p_K = 0$, $p_k = 3, 5$ *and every finite non-cyclic subgroup* $H \subset \Gamma$ *with* $p_k \mid \#H$ *is a dihedral group.*
(iii) $p_K = 0$, $p_k = 2$ *and every maximal finite, non-cyclic subgroup of* Γ *is isomorphic to either* A_4 *or* D_{2^s} *for some* $s \geqslant 1$.

We note that this assumption on Γ implies that any two maximal finite, non-cyclic, non-conjugated subgroups H, H' of Γ have either intersection $\{1\}$ or their intersection is a maximal cyclic subgroup of both H and H'.

Proof. As in the proof of 3.6 one shows that it suffices to consider Γ's of the form $G_1 *_{G_3} G_2$. For these groups the statements follow from the properties of the finite subgroups considered in Section 2 and the classification of the discontinuous groups of the form $G_1 *_{G_3} G_2$, given in Section 3. $\square$

Definition 4.3 (The graph $\mathcal{T}^c$). The group Γ is supposed to be ordinary. The collection of the maximal finite subgroups of Γ is denoted by $\max \Gamma$. We associate to Γ a graph $\mathcal{T}^c$ on which the group Γ acts.

The *vertices* of the graph $\mathcal{T}^c$ are the following finite subgroups $H \subset \Gamma$:

(v1) $H \in \max \Gamma$.
(v2) If $p_K = p = 2$ or 3, then a p-cyclic subgroup H of Γ is a vertex if:
  (a) If $H \subset H' \in \max \Gamma$, then H' is not of Borel type.
  (b) The group H is contained in at least two elements of $\max \Gamma$.

The *edges* $\{v_1, v_2\}$ of $\mathcal{T}^c$ are defined by

(e1) There exist lattice classes $[M_1]$, $[M_2]$ with stabilizers v_1 and v_2, such that for every $[M] \in [[M_1], [M_2]]$ the stabilizer $\Gamma_{[M]}$ is contained in v_1 or v_2.
(e2) In case $p_K = p = 2$ we *exclude* that the triple $(v_1, v_1 \cap v_2, v_2)$ is isomorphic to $(D_\ell, C_2, D_{\ell'})$.
(e3) In case $p_K = p = 3$ we *exclude* that the triple $(v_1, v_1 \cap v_2, v_2)$ is isomorphic to $(\mathrm{PSL}_2(\mathbf{F}_3), C_3, \mathrm{PSL}_2(\mathbf{F}_3))$.

4.3.1. Comments on (e1)

We note that there is a rather subtle point in the formulation of (e1). One would like to state that $[M_1]$, $[M_2]$ are vertices of $\mathcal{T}$. This is true except for the cases $p_K = p = 2$ or 3, $H \cong C_p$ satisfying (v2) and such that H is contained in precisely two maximal finite subgroups H_1, H_2 of Γ. In this situation (e1) prescribes that $\{H_1, H\}$ and $\{H, H_2\}$ are edges of $\mathcal{T}^c$. Indeed, let $[N_i]$ denote the unique H_i-invariant lattice class for $i = 1, 2$. Then $[N_i]$, $i = 1, 2$, are vertices of $\mathcal{T}$. It is not difficult to show that there exists a lattice class $[M]$ in $[[N_1], [N_2]]$ with stabilizer H. Thus $\{H_1, H\}$ and $\{H, H_2\}$ are edges of $\mathcal{T}^c$. However $[M]$ need not be a vertex of $\mathcal{T}$. In view of this one may replace (e1) by the condition that one of the two lattice classes $[M_1]$, $[M_2]$ belongs to $\mathcal{T}$.

In order to see that property (e1) of an edge $e = \{v_1, v_2\}$ is natural, we will show that $H := v_1 \cap v_2$ is non-trivial and that the homomorphism $v_1 *_H v_2 \to \Gamma$ is injective. We may of course suppose that $v_1 \not\subset v_2$ and $v_2 \not\subset v_1$. Every lattice class $[M] \in [[M_1], [M_2]]$ has a non-trivial stabilizer $\Gamma_{[M]}$, since Γ is indecomposable. Moreover this group is contained in either v_1 or v_2. It suffices to produce $[M] \in [[M_1], [M_2]]$ with $\Gamma_{[M]}$ contained in both v_i. Indeed, then $1 \neq \Gamma_{[M]} = v_1 \cap v_2$, and we can apply part (1) of Theorem 3.3.

Suppose that for no $[M] \in [[M_1], [M_2]]$ the group $\Gamma_{[M]}$ is contained in both v_i. For any $g \in v_1$, $g \notin v_2$ one defines $[M_1(g)] \in [[M_1], [M_2]]$ by: for $[M] \in [[M_1], [M_2]]$ one has $g \in \Gamma_{[M]}$ if and only if $[M] \in [[M_1], [M_1(g)]]$. Let $[[M_1], [M_1^*]]$ denote the union of all such $[[M_1], [M_1(g)]]$. Then $[M] \in [[M_1], [M_2]]$ has the property $\Gamma_{[M]} \subset v_1$ if and only $[M] \in [[M_1], [M_1^*]]$. There is a $[M_2^*] \in [[M_1], [M_2]]$ with the similar property w.r.t. v_2. These two segments cover $[[M_1], [M_2]]$ and have empty intersection. This is a contradiction.

4.3.2. Further comments on the definition

Part (v1) of the definition is natural, too. The additions (v2), (e2) and (e3) have their origin in the special features for $p_K = p = 2, 3$ (see 3.15). In particular, if one omits the extra vertices of (v2), then $\mathcal{T}^c$ will in general have cycles and will not be a tree.

We remark moreover that no two maximal finite subgroups $H_1 \neq H_2$ in part (b) of (v2) are conjugated in Γ. This can be seen as follows: Let $H_1 \neq H_2$ be two maximal finite subgroups of Γ containing $H \cong C_p$. Write $[M_i]$, $i = 1, 2$, for the unique invariant lattice class of the group H_i. In general, property (e1) does not hold for the segment $[[M_1],[M_2]]$. However this segment has a subdivision in segments $[[N_i],[N_{i+1}]]$ such that the stabilizer T_i of each $[N_i]$ in Γ is a maximal finite subgroup and each $[[N_i],[N_{i+1}]]$ satisfies condition (e1). According to 4.3.1, $T_i *_{T_i \cap T_{i+1}} T_{i+1}$ is a realizable amalgam. Each group T_i is a maximal finite subgroup of Γ containing $H \cong C_p$. Condition (v2), combined with Proposition 3.9 yields that $T_i *_{T_i \cap T_{i+1}} T_{i+1}$ is $D_\ell *_{C_2} D_\ell$ if $p = 2$ and is equal to $\mathrm{PSL}_2(\mathbf{F}_3) *_{C_3} \mathrm{PSL}_2(\mathbf{F}_3)$ for $p = 3$. Hence each maximal finite subgroup of Γ containing H is a D_ℓ for $p = 2$ or a $\mathrm{PSL}_2(\mathbf{F}_3)$ for $p = 3$. Let $B \subset \mathrm{PGL}_2(K)$ be the unique Borel group containing H. Then there exists a maximal finite subgroup, say H_3, of Γ containing $\Gamma \cap B$. Since H_3 is either a D_ℓ or a $\mathrm{PSL}_2(\mathbf{F}_3)$, one concludes that $\Gamma \cap B = H$.

Now suppose that there exists $\gamma \in \Gamma$ with $\gamma H_1 \gamma^{-1} = H_2$. According to Proposition 2.1, the morphism $\mathbf{P}^1 \to \mathbf{P}^1/H_i$ has two branch points and precisely one of them is wild. Thus H_i contains a wild ramification group $R_i \supset H$ and every wild ramification group in H_i is H_i-conjugated with R_i. Then $R_i = H_i \cap B$. Now $\gamma R_1 \gamma^{-1}$ is a wild ramification group of H_2 and is therefore equal to $\delta R_2 \delta^{-1}$ for some $\delta \in H_2$. Then $\tilde{\gamma} := \delta^{-1}\gamma \in \Gamma$ satisfies $\tilde{\gamma} H_1 \tilde{\gamma}^{-1} = H_2$ and $\tilde{\gamma} R_1 \tilde{\gamma}^{-1} = R_2$. Hence $\tilde{\gamma} \in B \cap \Gamma = H$. This yields the contradiction $H_1 = H_2$.

4.3.3. The action of Γ on the vertices of $\mathcal{T}^c$

This action is defined by $\gamma(v) = \gamma H \gamma^{-1}$ for $v = H$ a vertex and $\gamma \in \Gamma$. We write Γ_v for $\{\gamma \in \Gamma \mid \gamma H \gamma^{-1} = H\}$. Suppose that $v = H$ is a maximal finite subgroup of Γ. If H is not of Borel type then it has a unique separating lattice class $[M]$. The group Γ_v stabilizes $[M]$ and is therefore finite and hence $\Gamma_v = H$. If H is of Borel type then it is contained in a unique Borel group B and Γ_v is easily seen to be $\Gamma \cap B$. The latter group is an increasing union of its finite subgroups. Since Γ is finitely generated we conclude that $\Gamma \cap B$ is in fact a finite group. By the maximality of H we have again that $\Gamma_v = H$.

Suppose now that $v = H$ is not maximal. Then $p_K = p = 2, 3$ and $H \cong C_p$ satisfies (a) and (b) of (v2). As before, Γ_v is equal to the finite group $\Gamma \cap B$ where B is the unique Borel subgroup which contains H. Choose two maximal finite subgroups $H_1 \neq H_2$ containing H. We may suppose that $H_1 \supset \Gamma_v$. Let $[M_1]$, $[M_2]$ denote the unique invariant lattice classes for H_1, H_2. Suppose that H_1, H_2 satisfy property (e1), then $H_1 *_{H_1 \cap H_2} H_3 \subset \Gamma$. Since $H \subset H_1 \cap H_2$, we conclude by Proposition 3.9 that $H_1 = D_\ell$ for $p = 2$ and $H_1 = \mathrm{PSL}_2(\mathbf{F}_3)$ for $p = 3$, because $H \subset H_1 \cap H_3$. By construction $\Gamma_v \subset H_1 \cap B$. The group $H_1 \cap B$ is easily seen to be C_p. Thus $\Gamma_v = H$.

In the opposite situation, one considers $[M] \in [[M_1],[M_2]]$, closest to $[M_1]$ such that $\Gamma_{[M]}$ is not contained in any of the H_i. Let H_3 be a maximal finite subgroup of Γ containing $\Gamma_{[M]}$ and let $[M_3]$ denote its unique invariant lattice class. Then H_1, H_3 satisfy property (e1). As before this implies that $\Gamma_v = H$.

The aim of this section is to show that the graph $\mathcal{T}^c$ is actually *a locally finite tree and that $\mathcal{T}^c/\Gamma$ is a contracted, indecomposable finite tree of groups*.

Lemma 4.4. *The notions and notations are those of Section* 2.1. *Let the set of lattice classes $\mathcal{M}$ consists of finitely many Γ-orbits. Then $\mathcal{M}$ is discrete and Γ acts on the tree $\mathcal{T}_{\mathcal{M}}$. Moreover,*

(i) *The ends of the tree $\mathcal{T}_M$ are in bijective correspondence with the points of the limit set $\mathcal{L}$ for the group Γ.*
(ii) *For $x \in \Omega$ there exists a unique equivalence class $[M] \in \mathrm{conv}(\mathcal{M})$ such that $\mathrm{red}_{[M]}(x) \neq \mathrm{red}_{[M]}(e)$ for all edges $e \in \mathcal{T}_{\mathcal{M}\cup\{[M]\}}$ which have $[M]$ as vertex.*

Proof. The discreteness of $\mathcal{M}$ and (i) follow easily from Section 2.1. Write T for $\mathcal{T}_M$. The reduction map $\mathrm{red}_T : \Omega \to \overline{(\Omega, T)}$ maps x to some point $\mathrm{red}_T(x)$. If $\mathrm{red}_T(x)$ lies on a single irreducible component of $\overline{(\Omega, T)}$, then the corresponding lattice class $[M]$, which is a vertex of $\mathcal{T}_{\mathcal{M}}$, has the required property. Suppose that $\mathrm{red}_T(x)$ is a double point, lying on the intersection of two irreducible components L_{v_1}, L_{v_2} corresponding to vertices v_1, v_2 of $\mathcal{T}_{\mathcal{M}}$. Then one chooses points $y_1, y_2 \in \Omega$ with images on the non singular points of L_{v_1} and L_{v_2}. Let $[M]$ denote the lattice class given by the triple y_1, x, y_2. One easily sees that $[M] \in [[M_1], [M_2]]$ and that $[M]$ is the unique lattice class with the required property. □

Definition 4.5. Let B be a Borel subgroup of $\mathrm{PGL}_2(K)$ and let $x_B \in \mathbf{P}^1_K$ be the unique point that is fixed by B. We suppose that x_B is not a limit point for Γ, i.e., $x_B \notin \mathcal{L}$. We recall that $\mathcal{T}$ is a subdivision of the tree $\mathcal{T}_{\mathcal{L}}$ (possibly needed in order to let Γ act without inversions). Let $\mathcal{M}(T)$ be the set of equivalence classes corresponding to the vertices of $\mathcal{T}$. We say that an equivalence class $[M] \in \mathrm{conv}(\mathcal{M}(T))$ is *the equivalence class that is closest to B* if $[M]$ has property (ii) of Lemma 4.4 with respect to $x = x_B$.

Lemma 4.6. *Let $x \in \mathbf{P}^1_K$ be a fixed point of some element in Γ of finite order. Let $G_x \subset \Gamma$ denote the stabilizer of x and $H_x \subset G_x$ the maximal finite subgroup of G_x. Then*

(1) *$x \in \Omega$ if and only if $H_x = G_x$.*
(2) *If $p_K = 0$, then H_x is a maximal finite cyclic subgroup of Γ.*
(3) *If $p_K = p > 0$ and p divides $\#H_x$, then $x \in \Omega$ and $H_x = G_x = \Gamma \cap B$, where B is the Borel group associated to x.*

Proof. The group G_x is a discontinuous subgroup of a Borel group. If $p_K = 0$, then G_x does not contain unipotent elements $\neq 1$ and this implies that G_x has the form $\{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mid a \in A\}$ and $A \subset K^*$ a certain discontinuous subgroup. The group H_x is then equal to $\{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mid a \in B\}$, where B is a finite subgroup of K^*. Thus H_x is cyclic. If $p_K = p > 0$ and G_x does not contain unipotent elements $\neq 1$, then again H_x is a cyclic group of order not divisible by p. If $p_K = p > 0$ and G_x does contain a unipotent element $\neq 1$ (or equivalently G_x contains an element of order p), then one easily sees that G_x is the filtered union of finite groups. Since Γ is finitely generated, this implies that G_x itself is finite.

(1) If $x \in \Omega$ then clearly G_x is finite. Suppose that $x \notin \Omega$. We fix an embedding of $T_\Gamma := \mathcal{T}/\Gamma$ in $\mathcal{T}$. After replacing x by a Γ-conjugate we may suppose that H_x stabilizes a vertex v of T_Γ. Then H_x also stabilizes the half line L starting with v in the "direction" x. Infinitely many Γ-conjugates of some vertex w of T_Γ lie on the half line L. Let G denote the stabilizer of w. Then for infinitely many $\gamma \in \Gamma$ one has $\gamma H_x \gamma^{-1} \subset G$. This has as consequence that the group $\{\gamma \in \Gamma \mid \gamma h \gamma^{-1} = h \text{ for all } h \in H_x\}$ is infinite. Then also G_x is infinite. The reasoning above also implies statements (2) and (3). □

Definition 4.7 (*The tree* $\mathcal{T}^\dagger$). The graph $\mathcal{T}^c$ is an abstract graph on which the group Γ acts. However, one associates to each vertex $v \in \mathcal{T}^c$ a lattice class $[M_v]$ as follows:

(i) If v is not of Borel type, then $[M_v]$ is the unique separating lattice class for the group v.
(ii) If v is a subgroup of a Borel subgroup $B \subset \mathrm{PGL}_2(K)$, then according to 4.6 the fixed point x of B lies in Ω and one defines $[M_v]$ to be the unique lattice class in $\mathrm{conv}(\mathcal{M}(\mathcal{T})$ that is closest to B.

Put $\mathcal{M}(\mathcal{T}^c) = \{[M_v] \mid v \text{ vertex of } \mathcal{T}^c\}$. We define $\mathcal{T}^\dagger := \mathcal{T}_{\mathcal{M}(\mathcal{T}^c)}$. By $\mathcal{M}(\mathcal{T}^\dagger)$ we denote the set of lattice classes corresponding to the vertices of the tree $\mathcal{T}^\dagger$. We recall that $\mathcal{M}(\mathcal{T}^\dagger) = \mathcal{M}(\mathcal{T}^c) \cup V(\mathrm{conv}(\mathcal{M}(\mathcal{T}^c))$.

Lemma 4.8. *Let $v = H$ be a vertex of $\mathcal{T}^c$ and $[M_v]$ its associated lattice class. Then $v = \Gamma_v$ coincides with the stabilizer of the lattice class $[M_v]$.*

Proof. For any $\gamma \in \Gamma$ one has $\gamma v := \gamma H \gamma^{-1}$ and $[M_{\gamma v}] = [\gamma M_v]$. Therefore Γ_v is a subgroup of the stabilizer $\widetilde{\Gamma}_v$ of $[M_v]$. If H is a maximal finite subgroup then clearly $\widetilde{\Gamma}_v = \Gamma_v = H$.

Suppose that $p_K = p = 2, 3$ and $H \cong C_p$. Let B denote the unique Borel group containing H and $x_B \in \Omega$ the fixed point of this Borel group. Let H_1, H_2 denote two, non-conjugated, maximal finite subgroups of Γ containing H and let $[M_1]$, $[M_2]$ denote their separating lattices. If $[M_1] = [M_v]$ then the images of x_B and $[M_2]$ in $\mathbf{P}(M_v \otimes_{K^0} k)$ are distinct. If $[M_2] = [M_v]$ then the images of x_B and $[M_1]$ in $\mathbf{P}(M_v \otimes_{K^0} k)$ are distinct. If $[M_v] \neq [M_1], [M_2]$, then the images of x_B, $[M_1]$, $[M_2]$ in $\mathbf{P}(M_v \otimes_{K^0} k)$ are distinct. Thus in all cases H has at least two fixed points in $\mathbf{P}(M_v \otimes_{K^0} k)$. Thus H acts trivially on $\mathbf{P}(M_v \otimes_{K^0} k)$. By Proposition 2.2, the group $\widetilde{\Gamma}_v$ is of Borel type.

Suppose that $\widetilde{\Gamma}_v \neq H = \Gamma_v$. Let $H_3 \supset \widetilde{\Gamma}_v$ be a maximal finite subgroup of Γ. As in 4.3, one concludes for that H_3 is D_ℓ (with odd ℓ) for $p = 2$ and $H_3 = \mathrm{PSL}_2(\mathbf{F}_3)$ for $p = 3$. In the second case $\widetilde{\Gamma}_v = \mathrm{PSL}_2(\mathbf{F}_3)$ and is a maximal finite subgroup of Γ. In the first case one finds that $\widetilde{\Gamma}_v = D_m$ for some divisor m of ℓ. In both cases $\widetilde{\Gamma}_v$ is not of Borel type, which yields a contradiction. Therefore $\widetilde{\Gamma}_v = \Gamma_v = H$. $\square$

Theorem 4.9. *Let Γ be ordinary. Then the graph $\mathcal{T}^c$ is a tree.*

Proof. Suppose that $\mathcal{T}^c$ contains three vertices $\{v_1, v_2, v_3\}$ such that $\{v_i, v_j\}$ is an edge for all $i < j$. For convenience we write also v_i for the lattice class $[M_{v_i}]$ in $\mathcal{T}^\dagger$ and Γ_i for the stabilizer of v_i. The minimal subtree T of $\mathcal{BT}$ containing $\{v_1, v_2, v_3\}$ has vertices $\{v, v_1, v_2, v_3\}$ and edges $\{v, v_i\}$ for $i = 1, 2, 3$. The vertex v is defined as $[v_1, v_2] \cap [v_1, v_3]$. By construction $T \subset \mathcal{T}^\dagger$. Let S be the stabilizer of v in Γ. For $i < j$ one has $S \subset \Gamma_i$ or $S \subset \Gamma_j$, and $S \supset \Gamma_i \cap \Gamma_j \neq 1$, since $\{v_i, v_j\}$ is an edge. *Suppose that S is contained in every* Γ_i then $S = \Gamma_i \cap \Gamma_j$ for all $i < j$ and $S = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$. *Suppose that S is not contained in, say,* Γ_1, then $S \subset \Gamma_2 \cap \Gamma_3$ and consequently $S = \Gamma_2 \cap \Gamma_3$. From $S \supset \Gamma_1 \cap \Gamma_2$ we conclude that $H := \Gamma_1 \cap \Gamma_2 \cap \Gamma_3 \neq 1$. If H contains an element of order not divisible by p_k, then the vertices v_1, v_2, v_3 lie on a line in $\mathcal{BT}$, namely the axis of that element. We conclude that the order of H is a power of p_k.

Consider the case $p_K = 0$, $p_k > 2$. Since Γ is ordinary, every Γ_i is a dihedral group, H is a cyclic group and H is normal in every Γ_i. From the description in Section 2 of the tree of D_ℓ where $p_k \mid \ell$, one concludes that the unique separating lattice class for D_ℓ lies on the axis of any non-trivial subgroup of $C_\ell \subset D_\ell$. One finds the contradiction that v_i, v_2, v_3 lie on a segment in $\mathcal{BT}$.

Consider the case $p_K = 0$, $p_k = 2$. If every $\Gamma_i \cong A_4$, then $H = \Gamma_1 \cap \Gamma_2 \cap \Gamma_3$ is isomorphic to C_2. Let $L \subset \mathcal{BT}$ denote the H-axis. From the description of the tree of A_4 in Section 2.5, one concludes that the lattice classes v_1, v_2, v_3 are not on L. Their projections $\mathrm{pr}_L(v_1)$, $\mathrm{pr}_L(v_2)$, $\mathrm{pr}_L(v_3)$ are distinct, since for $i < j$ the subgroup generated by Γ_i and Γ_j is equal to $\Gamma_i *_H \Gamma_j$. Moreover, the stabilizer of $\mathrm{pr}_L(v_i)$ contains the unique subgroup of v_i, which is isomorphic to D_2. Suppose that $\mathrm{pr}_L(v_2) \in [\mathrm{pr}_L(v_1), \mathrm{pr}_L(v_3)]$. Then $[v_1, v_3]$ contains $\mathrm{pr}_L(v_2)$ and the stabilizer of this lattice class is not contained in v_1 or v_3. This contradicts the assumption that $\{v_1, v_3\}$ is an edge of $\mathcal{T}^c$. Similar arguments rule out all the other possible situations for $p_K = 0$, $p_k = 2$.

Consider the case $p_K = p > 0$. For $i < j$ the group $\Gamma_i *_{\Gamma_i \cap \Gamma_j} \Gamma_j$ is a subgroup of Γ and thus a realizable amalgam. Moreover H is a p-group. According to Corollary 3.11 this leaves only the possibilities $p = 3$, $H = C_3$ and each $\Gamma_i \cong \mathrm{PSL}_2(\mathbf{F}_3)$ or $p = 2$, $H = C_2$ and each Γ_i is a dihedral group D_{ℓ_i} with odd ℓ_i. This is excluded by (e2) and (e3) of Definition 4.3.

Now we consider the case where $\mathcal{T}^c$ contains a "circle" with consecutive vertices $v_1, v_2, \ldots, v_s, v_1$ with $s > 3$ and s minimal. We will use a "cyclic" notation for the vertices, i.e., $v_{i+s} = v_i$ for all $i \in \mathbf{Z}$. Let $T \subset \mathcal{BT}$ denote the smallest tree containing $v_1, \ldots, v_s$. By construction $T \subset \mathcal{T}^\dagger$. For convenience we will write $\Gamma_i := \Gamma_{v_i}$. First we will show that the extremal vertices of T are precisely $\{v_1, \ldots, v_s\}$.

A vertex $v \notin \{v_1, \ldots, v_s\}$ of T has at least three edges. Hence every extremal vertex of T is some v_i. Suppose that some v_b is not an extremal edge of T. Then there are extremal edges v_a, v_c of T such that v_b lies in the segment $[v_a, v_c]$. We may suppose that $1 \leqslant a < b < c \leqslant s$. Let $\mathrm{pr}: T \to [v_a, v_c]$ denote the projection. This means that $\mathrm{pr}(v_d)$ is the point $[v_a, v_d] \cap [v_d, v_c]$. For neighbours v_d, v_{d+1} with $b \neq d, d+1$ the point v_b is not contained in $[\mathrm{pr}(v_d), \mathrm{pr}(v_{d+1})]$. Suppose the opposite, then v_b lies on $[v_d, v_{d+1}]$ and Γ_b is contained in either Γ_d or Γ_{d+1}. Then Γ_b is not a maximal finite subgroup and we are in the situation $\Gamma_b = C_p$ with $p_K = p = 2$ or 3. Then $\Gamma_d \cap \Gamma_{d+1} = C_p$ and by 3.11, this only holds if $p = 2$ and one of the groups Γ_d, Γ_{d+1} is equal to $B(n, 1)$ with $n > 1$. The latter is excluded by part (v2) of Definition 4.3.

One concludes that $\mathrm{pr}(v_d) \in [v_a, v_b]$ for d with $a \leqslant d \leqslant b$ or $c < d \leqslant a + s$. The same reasoning yields that $\mathrm{pr}(v_d) \in [v_b, v_c]$ if $b \leqslant d < a + s$. Therefore $\mathrm{pr}(v_d) = v_b$ for d with $c < d < a + s$. If $c < a + s - 1$, then v_b lies on $[v_{a+s-1}, v_{a+s}]$. If $c = a + s - 1$, then v_b lies on $[v_c, v_a]$. In both cases one finds a contradiction as above. We conclude that the extremal edges of T are $\{v_1, \ldots, v_s\}$.

In the following we view the tree T as a topological space by identifying each edge with a copy of $[0, 1] \subset \mathbf{R}$. It is convenient to assume that $|K^*| = \mathbf{R}_{>0}$ (this can be achieved by replacing K be a larger complete, algebraically closed extension). Then all the points on the topological tree T correspond with lattice classes. For $i \in \{1, \ldots, s\}$ one considers the subset T_i^* of T corresponding to lattice classes $[M]$ such that its stabilizer $\Gamma_{[M]}$ is contained in Γ_i. The complement of T_i^* is the union over the elements $g \in \bigcup_{j=1}^{s} \Gamma_j$, $g \notin \Gamma_i$ of the set

$V(g) = \{[M] \in T \mid g \in \Gamma_{[M]}\}$. Since each $V(g)$ is closed, one has that T_i^* is open. Let T_i be the connected component of T_i^* containing v_i. The sets T_i have the following properties:

(1) T_i is convex in the sense that with $a, b \in T_i$ also $[a, b] \subset T_i$.
(2) $[v_i, v_{i+1}] \subset T_i \cup T_{i+1}$ and $[v_i, v_{i+1}] \cap T_i \cap T_{i+1}$ is not empty.
(3) $\bigcup_i T_i = T$ since every point of T lies on $[v_i, v_{i+1}]$ for some i.

If the intersection $T_{i_0,i_1,\ldots,i_t}$ of the $T_{i_0}, \ldots, T_{i_t}$ is non-empty then it is convex. Suppose that each T_i meets only T_{i-1} and T_{i+1}. Then the union of the segments $[v_i, v_{i+1}]$ produces a circle in T, which is impossible. Hence there are $i < j$ with $T_i \cap T_j \neq \emptyset$ and such that $\{v_i, v_j\}$ is not an edge. It follows that $[v_i, v_j] \subset T_i \cup T_j$ and that $\Gamma_i *_{\Gamma_i \cap \Gamma_j} \Gamma_j$ is either $D_\ell *_{C_2} D_{\ell'}$ with odd ℓ, ℓ' and $p_K = p = 2$ or $\mathrm{PSL}_2(\mathbf{F}_3) *_{C_3} \mathrm{PSL}_2(\mathbf{F}_3)$ and $p_K = p = 3$.

We consider first the case $p_K = p = 2$. Let $\mathrm{pr}: T \to [v_i, v_j]$ denote the projection of the tree T on the segment $[v_i, v_j] \subset T$. It follows from the structure of the subgroups of dihedral group that for any $[M] \in [v_i, v_j]$ with $[M] \neq v_i, v_j$ that $\Gamma_{[M]} = C_2$. Now $\mathrm{pr}(v_{i+1}) \neq v_i, v_j$ and one concludes that $\Gamma_{i+1} \cong B(n, 1)$. Similarly $\Gamma_{i-1} \cong B(n', 1)$. The two groups Γ_{i-1}, Γ_{i+1} lie in the same Borel subgroup of $\mathrm{PGL}_2(K)$ and generate a finite subgroup of Γ. Therefore one of the two groups, say Γ_{i+1}, is not maximal and, by (v2), isomorphic to C_2. Finally $\Gamma_{i-1} \cong B(n', 1)$ and $n' > 1$ is in conflict with (v1) and (v2). The proof for the case $p_K = p = 3$ is completely similar. □

Theorem 4.10. *Suppose that $p_K = 0$. The tree $\mathcal{T}^c$ has properties* (i)–(iii) *below. As a consequence the tree of groups $T^c := \mathcal{T}^c/\Gamma$, obtained by embedding T^c in $\mathcal{T}^c$, has the properties of Theorem* 3.12.

(i) *The map $v \mapsto \Gamma_v$, from the set of vertices of $\mathcal{T}^c$ to the set of maximal finite subgroups of Γ, is a bijection.*
(ii) *The stabilizer of an edge $e \in \mathcal{T}^c$ is a maximal finite cyclic subgroup of Γ.*
(iii) *Let $v \in \mathcal{T}^c$ be a vertex. Then the following two statements hold*:
  (a) *For any maximal cyclic subgroup $H \subset \Gamma_v$, there are at most two edges e with vertex v, such that $\Gamma_e = H$.*
  (b) *Suppose that $p_k > 2$. If two distinct edges $e' = \{v, v'\}$, $e'' = \{v, v''\}$ have the same stabilizer $H \subset \Gamma_v$, then* $\mathrm{red}_v(e') \neq \mathrm{red}(e'')$.

Proof. (i) follows from the definition of $\mathcal{T}^c$ and (ii) is proved in Corollary 3.6.

(iii) Let H be a maximal cyclic subgroup of Γ_v. Suppose that $p_k \nmid \#H$. The group H acts faithfully on $\mathbf{P}(M_v \otimes k)$ and has there two fixed points which are the images $\bar{a}, \bar{b}$ of the two fixed points a, b of H on $\mathbf{P}^1(K)$. An edge $e' = \{v, v'\}$ with $\Gamma_{e'} = H$ has the property that $\psi_v(e') := \psi_v([M_{v'}])$ is invariant under H and thus $\psi_v([M_{v'}]) = \bar{a}$ or $\bar{b}$. Suppose that two edges $e' = \{v, v'\}$, $e'' = \{v, v''\}$ have $\Gamma_{e'} = \Gamma_{e''} = H$ and $\psi_v([M_{v'}]) = \psi([M_{v''}]) = \bar{a}$. Then the three separating lattices $[M_v]$, $[M_{v'}]$, $[M_{v''}]$ lie on the axis of H in $\mathcal{BT}$ and $[M_v]$ does not lie in between $[M_v']$ and $[M_{v''}]$. Thus, say, $[M_{v'}]$ lies in between $[M_v]$ and $[M_{v''}]$. This contradicts the assumption that $\{v, v''\}$ is an edge.

Suppose that the order of H is divisible by p_k and $p_k > 2$. Then $\Gamma_v \cong D_\ell$ with $p_k|\ell$ and $H \cong C_\ell$. Let again $e' = \{v, v'\}$ be an edge. Then $\Gamma_{v'}$ is also isomorphic to D_ℓ. Moreover the separating lattices $[M_v]$, $[M_{v'}]$ lie an the axis of H. The latter follows from the picture in Section 2, concerning the groups D_ℓ with $p_k|\ell$. For a second edge $e''=\{v,v''\}$ the same holds. Moreover, as above, $[M_v]$ must lie in between $[M_{v'}]$ and $[M_{v''}]$ on the axis of H.

Suppose that $p_k=2$ and $\#H=2^s$ with $s>1$. Every maximal finite subgroup w of Γ, containing H, is isomorphic to D_{2^s}. Let $L\subset\mathcal{BT}$ denote the axis of H. For maximal finite subgroups $w_1\neq w_2$ of Γ, containing H, one has that $w_1 *_H w_2\to\Gamma$ is an injection. This implies that the projections $\mathrm{pr}_L([M_{w_1}])$, $\mathrm{pr}_L([M_{w_2}])$ are distinct. As a consequence there are at most two edges e,e' of w with $H=G_e=G_{e'}$. In this case however, $\mathrm{red}_v(e)=\mathrm{red}_v(e')$.

Suppose $p_k=2$ and $H\cong C_2$. Let $L\subset\mathcal{BT}$ denote the axis of H. For any maximal finite subgroup w of Γ, which has H as maximal cyclic subgroup, one has that $[M_w]$ does not lie on L. Moreover, for two such groups $w_1\neq w_2$, the argument above shows that the projections $\mathrm{pr}_L([M_{w_1}])$, $\mathrm{pr}_L([M_{w_2}])$ are distinct. As a consequence there are at most two edges e,e' of v with $H=G_e=G_{e'}$. Again in this situation $\mathrm{red}_v(e)=\mathrm{red}_v(e')$.

Clearly, T^c has the properties (a)–(e) of Theorem 3.12. Suppose that $e_i=\{v_i,v\}$, $i=1,2$, are two edges in T^c with $\Gamma_{e_i}=H\subset\Gamma_v$. If $g\in\Gamma_v$ satisfies $gHg^{-1}=H$ and $g\notin H$, then $g(e_1)=e_2$. This contradicts the definition of T^c and we conclude that (f) of Theorem 3.12 is also valid. We note that property (iii)(b) has not been used here. □

Theorem 4.11. *Suppose that $p_K=p>0$. The tree $\mathcal{T}^c$ satisfies the properties* (1)–(5) *below. As a consequence, the tree of groups $T^c:=\mathcal{T}^c/\Gamma$ satisfies the properties of Theorem* 3.14.

(1a) *The map $v\mapsto\Gamma_v$ is a bijection between the vertices of $\mathcal{T}^c$, with $\Gamma_v\not\cong C_p$, and the maximal finite subgroups of Γ.*
(1b) *Only for $p=2,3$, the group Γ may contain a maximal finite subgroup G which is a p-group. In that case $G\cong B(n,1)$ with $n>1$.*
(1c) *Only for $p=2,3$, the tree $\mathcal{T}^c$ may contain vertices v with $\Gamma_v\cong C_p$. In which case, the map $v\mapsto\Gamma_v$ yields a bijection between the vertices v with Γ_v not a finite maximal subgroup of Γ, and the subgroups $H\subset\Gamma$ satisfying* (v2) *of Definition* 4.3.
(2a) *The stabilizer Γ_e of an edge $e=\{v_1,v_2\}$ of $\mathcal{T}^c$ is a non-trivial group of Borel type.*
(2b) *If Γ_e is a p-group then $\Gamma_e\cong C_p$ and $p=2,3$. Furthermore, after interchanging v_1, v_2 if necessary, one has that Γ_{v_1} is a p-group. If $p=2$, then $\Gamma_{v_2}\cong D_\ell$ with odd ℓ. If $p=3$, then $\Gamma_{v_2}\cong\mathrm{PSL}_2(\mathbf{F}_3)$.*
(2c) *If Γ_e contains a p-group, then Γ_{v_i} is of Borel type for precisely one i.*
(3) *Let $v\in\mathcal{T}^c$ be a vertex such that the stabilizer Γ_v of v is not of Borel type and let e be an edge of v. Then the following holds*:
 (3a) *$\Gamma_e\subset\Gamma_v$ is a ramification group of the map $\varphi_{\Gamma_v}:\mathbf{P}^1_K\to\mathbf{P}^1_K/\Gamma_v$.*
 (3b) *If $e'\neq e$ is an edge of v and $\Gamma_e=\Gamma_{e'}$, then* $\mathrm{red}_v(e)\neq\mathrm{red}_v(e')$.
(4) *If the stabilizer Γ_v of a vertex $v\in\mathcal{T}^c$ is of Borel type and Γ_v is not a p-group, then Γ_v acts transitively on the edges e that contain v.*
(5) *If Γ_v is a p-group, then for all edges $e,e'\ni v$ one has $\Gamma_e=\Gamma_{e'}\cong C_p$. Furthermore, if $\Gamma_v\cong C_p$, then the vertex v is contained in at least two edges of $\mathcal{T}^c$.*

Proof. (1) follows from the construction of $\mathcal{T}^c$ and 3.11.

(2a) The group Γ_e is non-trivial because Γ is indecomposable and it is of Borel type since it stabilizes at least two lattice classes.

(2b), (2c) and (3a) follow from 3.11 and conditions (e2), (e3) of 4.3.

(3b) Let $e = \{v, v'\}$ be an edge such that $p \mid \#\Gamma_e$ and let B be the unique Borel group containing Γ_e. By (2c), $\Gamma_{v'}$ is of Borel type and actually equal to $\Gamma \cap B$. In particular, e is determined by Γ_e.

If $p \nmid \#\Gamma_e$, then Γ_e is a maximal cyclic group of order prime to p. The group Γ_e has to fixed points $a, b \in \mathbf{P}^1(K)$ and e is determined by Γ_e and a choice of one of the fixed points. One has to prove that the images of a, b in $\mathbf{P}(M_v \otimes k)$ are distinct. This follows from the classification in Proposition 2.1.

(4) Assume that Γ_v is of Borel type $B(n, m)$ with $m > 1$ and let $e = \{v, v'\}$ be an edge of $\mathcal{T}^c$. From Corollary 3.11 and (e2), (e3) of Definition 4.3, one can read off the possibilities for $\Gamma_v *_{\Gamma_e} \Gamma_{v'}$. Statement (iv) translates into: "the possible subgroups Γ_e are conjugated in $B(n, m)$."

The case $\Gamma_e \cong C_m$ is valid since all cyclic subgroups of order m in $B(n, m)$ are conjugated. The remaining possibilities for $\Gamma_v *_{\Gamma_e} \Gamma_{v'}$ are

(a) $B(n, q-1) *_{B(\mathbf{F}_q)} \mathrm{PGL}_2(\mathbf{F}_q)$ with $q \neq 2$.
(b) $B(n, \frac{q-1}{2}) *_{B(\mathbf{F}_q)} \mathrm{PSL}_2(\mathbf{F}_q)$ and $p \neq 2$.
(c) $B(n, 2) *_{D_3} A_5$ and $p = 3$.

Consider case (a) and let $e_1 = \{v, v_1\}$ be an edge. Let f_1, f_2 be a basis of K^2 over K, such that $\Gamma_{v_1} = \mathrm{PGL}(\mathbf{F}_q f_1 + \mathbf{F}_q f_2)$ and Γ_{e_1} is the Borel subgroup consisting of the elements which leave the line $\mathbf{F}_q f_1$ invariant. On this basis, the group $B(n, q-1)$ consists of the matrices $\{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbf{F}_q^*,\ b \in V\}$ where $V \subset K$ is a finite-dimensional vector space over $\mathbf{F}_q$. One has that $\mathbf{F}_q \subset V$ and $\mathbf{F}_q \neq V$. As in the proof of Proposition 3.10 one verifies that the condition that $\Gamma_v *_{\Gamma_{e_1}} \Gamma_{v_1}$ is a realizable amalgam is equivalent to $V = \mathbf{F}_q \oplus W$, where the $\mathbf{F}_q$-vector space W has the property $|w| > 1$ for every $w \in W$, $w \neq 0$. The lattice class $[M_v]$ in $\mathcal{T}^c$ associated to Γ_v is given by $M_v = K^0 \lambda f_1 + K^0 f_2$, where $\lambda \in K$ is chosen such that $|\lambda| = \max_{v \in V} |v|$. The lattice class $[M_{v_1}]$ associated to Γ_{v_1} is given by $M_{v_1} = K^0 f_1 + K^0 f_2$. One observes that the distance between the two lattice classes depends only on the group Γ_v. Let $e_2 = \{v, v_2\}$ be another edge of v. After conjugation with an element $g \in \Gamma_v = B(n, q-1)$ we may suppose that the subgroup $C_{q-1} = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbf{F}_q^*\}$ of $B(n, q-1)$ belongs to Γ_{v_2}. The intersection $\Gamma_v \cap \Gamma_{v_2}$ has the form $\{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbf{F}_q^*,\ b \in Z\}$, where Z is a 1-dimensional vector space over $\mathbf{F}_q$. Then $[M_{v_1}]$, $[M_{v_2}]$ lie on the axis of the group C_{q-1}, on the same side of $[M_v]$ and with the same distance to $[M_v]$. We conclude that $[M_{v_1}] = [M_{v_2}]$ and $v_1 = v_2$. This proves that Γ_v acts transitively on the edges of v in $\mathcal{T}^c$. The cases (b) and (c) can be handled in the same way.

(5) Γ_v is a $B(n, 1)$. For $n > 1$, the only possibilities for an edge $e = \{v, v'\}$ produce the amalgams $B(n, 1) *_{C_2} D_\ell$ with odd ℓ and $p = 2$ (see 3.11). By Definition 4.3, an edge $e = \{v, v'\}$ for the case $n = 1$ is only possible with $\Gamma_{v'} \cong D_\ell$ with odd ℓ and $p = 2$ or $\Gamma_{v'} \cong \mathrm{PSL}_2(\mathbf{F}_3)$ and $p = 3$.

Finally, the verification that T^c satisfies the properties of 3.14, is straightforward. We note that some of the stated properties of $\mathcal{T}^c$ are superfluous for this verification. □

Observations 4.12 (*Relations between the trees $\mathcal{T}$, $\mathcal{T}^c$ and $\mathcal{T}^\dagger$*). As before, we suppose that Γ is finitely generated, discontinuous, indecomposable, ordinary and its set of limit points $\mathcal{L}$ has more than two elements.

(1) *Γ acts on the tree $\mathcal{T}_{\mathcal{L}}$ without inversion and thus $\mathcal{T} = \mathcal{T}_{\mathcal{L}}$.*

$\mathcal{L}$ is also the set of equivalence classes of the ends of the tree $\mathcal{T}^c$. From 4.10 and 4.11 it follows that no inversion is possible.

(2) *The tree $\mathcal{T}^c$ has no extremal vertices.*

This follows from the descriptions 3.12 and 3.14 of $T^c := \mathcal{T}^c/\Gamma$, proved in 4.10 and 4.11.

(3) *$\mathcal{T}$ and $\mathcal{T}^\dagger$ have the "same" classes of infinite ends. Moreover every vertex of $\mathcal{T}$ is also a vertex of $\mathcal{T}^\dagger$.*

The infinite ends are, for both trees, in bijection with the limit points $\mathcal{L}$. A vertex $[M]$ of $\mathcal{T}$ is determined by three points of $\mathcal{L}$. The corresponding three ends of $\mathcal{T}^\dagger$ determine a vertex of $\mathcal{T}^\dagger$, which coincides with $[M]$.

(4) *$\mathcal{T} = \mathcal{T}^\dagger$ if and only if $[M_v]$ belongs to $\mathcal{T}$ for every vertex v of $\mathcal{T}^c$.*

(5) *If $p_K = p \geqslant 5$, then $\mathcal{T} = \mathcal{T}^\dagger$.*

One has to verify that $[M_v]$, attached to a maximal finite subgroup v of Γ, belongs to $\mathcal{T}$. If v is not of Borel type, then v has a unique invariant lattice in $\mathcal{BT}$. This lattice is equal to $[M_v]$ and is also equal to the fixed point of v on the tree $\mathcal{T}$. Let v be a Borel type $B(n,m)$, then $m > 1$. For a suitable basis e_1, e_2 of K^2, one can represent v by the collection of matrices $\{\left(\begin{smallmatrix}\zeta & a\\ 0 & 1\end{smallmatrix}\right) \mid \zeta \in \mathbf{F}_q^*,\ a \in A\}$, where $A \subset K$ is a finite-dimensional vector space over $\mathbf{F}_q$ and moreover $\max_{a\in A}|a| = 1$. For this representation, the fixed point of v is ∞. The invariant lattice classes can be represented by $M = K^0e_1 + K^0\lambda e_2$ with $\lambda \in K^*$ and $|\lambda| \leqslant 1$. For the lattice $M_1 := K^0e_1 + K^0e_2$ the image under $\mathrm{red}_{[M_1]}$ of the set $A \cup \{\infty\}$ of ramification points of $B(n,m)$ has at least three points. For the other lattices classes $[M]$, this image consists of the two points $\mathrm{red}_{[M]}\,0, \mathrm{red}_{[M]}\,\infty$. The vertex $[M_v]$ has at least two edges e_1, e_2 in $\mathcal{T}^\dagger$. The three points $\mathrm{red}_{[M_v]}\,e_1$, $\mathrm{red}_{[M_v]}\,e_2$, $\mathrm{red}_{[M_v]}\,\infty$ are distinct and lie in the image of the ramification points of v. This proves that $[M_v] = [M_1]$. The vertex v has at least three edges in $\mathcal{T}^c$. This implies that $\mathrm{red}_{[M_v]}\,\mathcal{L}$ consists of at least three points and $[M_v] \in \mathcal{T}$.

(6) *For $p_K = 2, 3$ one considers the set $F \subset \Omega$ consisting of the points x for which there exists a maximal finite subgroup $H \subset \Gamma$, which is a p-group, or a group $H \cong C_p$*

having property (v2) *of Definition* 4.3, *such that x is the fixed point of H. Then $\mathcal{T}^\dagger$ coincides with the tree $\mathcal{T}_{F\cup\mathcal{L}}$ defined in Section* 2.1.

This follows readily from the definition of $[M_v]$ for $v = H$ and H as above.

(7) *Let $p_K = 0$, $p_k > 2$ and let v be a maximal finite subgroup of Γ. If $v \ncong D_\ell$ with ℓ a power of p_k, then $[M_v]$ belongs to $\mathcal{T}$.*
Suppose that $v \cong D_\ell$ and ℓ is a power of p_k. We choose a coordinate z for $\mathbf{P}^1(K)$ such that v consists of the transformations

$$\left\{z \mapsto \zeta^a z^b \mid 0 \leqslant a < \ell,\ b = \pm 1\right\}$$

where ζ is a primitive ℓth-root of unity. The three lattice classes in $\mathcal{BT}$, invariant under v are $[M_1]$, $[M_v]$, $[M_{-1}]$, given by the ramification points $\{1, \zeta, \zeta^{-1}\}$, $\{1, -1, \infty\}$ and $\{-1, -\zeta, -\zeta^{-1}\}$. Let red_v *denote the reduction $\mathbf{P}^1(K) \to \mathbf{P}(M_v \otimes k)$. Then* $\mathrm{red}_v(\{\textit{edges of } v\})$ *is a subset of $\{1, -1, 0, \infty\}$, the images under* red_v *of the ramification points of v. Further,* $\mathrm{red}_v(\mathcal{L}) \subset \mathrm{red}_v(\{\textit{edges of } v\})$ *consists of Γ_v-orbits. There are the following possibilities*:

(a) $\mathrm{red}_v(\mathcal{L}) = \{1, -1, 0, \infty\}$. *Then $[M_1], [M_v], [M_{-1}] \in \mathcal{T}$.*
(b) $\mathrm{red}_v(\mathcal{L}) = \{1, -1\}$. *Then $[M_1], [M_{-1}] \in \mathcal{T}$, $[M_v] \notin \mathcal{T}$ and $[M_v] \in \mathcal{T}^\dagger$ is not an extremal vertex of $\mathcal{T}^\dagger$.*
(c) $\mathrm{red}_v(\mathcal{L}) = \{\delta, 0, \infty\}$ *with $\delta = \pm 1$. Then $[M_\delta], [M_v] \in \mathcal{T}$, $[M_{-\delta}] \notin \mathcal{T}^\dagger$.*
(d) $\mathrm{red}_v(\mathcal{L}) = \{0, \infty\}$. *Then $[M_1], [M_{-1}] \notin \mathcal{T}^\dagger$, $[M_v] \notin \mathcal{T}$ and $[M_v]$ is an extremal edge of $\mathcal{T}^\dagger$.*
(e) $\mathrm{red}_v(\mathcal{L}) = \{\delta\}$ *with $\delta = \pm 1$. Then $[M_\delta] \in \mathcal{T}$, $[M_v] \notin \mathcal{T}$, $[M_{-\delta}] \notin \mathcal{T}^\dagger$ and $[M_v]$ is an extremal vertex of $\mathcal{T}^\dagger$.*

The proof is a straightforward computation. One concludes that $\mathcal{T}^\dagger$ is obtained from $\mathcal{T}$ by possibly a subdivision of edges (occurs only in case (b)) and by possibly attaching extremal vertices (occurs only in cases (d) and (e)). The corresponding situation for $p_k = 2$ is somewhat different.

(8) *$\mathcal{T}^c$ is a contraction of $\mathcal{T}^\dagger$ in the following sense*:
(a) *$v \mapsto [M_v]$ is an injective map from the vertices of $\mathcal{T}^c$ to those of $\mathcal{T}^\dagger$.*
(b) *$\{v_1, v_2\}$ is an edge of $\mathcal{T}^c$ if and only if for every $[M] \in [M_{v_1}, M_{v_2}]$ the group $\Gamma_{[M]}$ is contained in v_1 or v_2 and no other $[M_v]$ lies in $[[M_{v_1}], [M_{v_2}]]$.*
(c) *$\mathcal{T}^c$ and $\mathcal{T}^\dagger$ have the "same" classes of infinite ends.*

This follows from the definition of $\mathcal{T}^\dagger$ and the fact that $\mathcal{T}^c$ is a tree.

(9) *In general, $\mathcal{T}^c \neq \mathcal{T}^\dagger$.*

This is illustrated by the example $\Gamma := \mathrm{PGL}_2(\mathbf{F}_q) *_{B(\mathbf{F}_q)} B(n, q-1)$ for $p \geqslant 5$.

5. Counting the number of branch points

In the sequel we will, unless otherwise stated, assume that Γ is a finitely generated, discontinuous, indecomposable, ordinary subgroup of $\mathrm{PGL}_2(K)$ *such that $\Omega/\Gamma \cong \mathbf{P}^1(K)$. Moreover we suppose that its set of limit points contains more than two points.*

For the counting of the number of branch points $\mathrm{br}(\Gamma)$ we will need to know the location of the ramification points in Ω, i.e., the points in Ω having a non-trivial stabilizer in Γ. In the previous section, a detailed description of the trees $\mathcal{T}$, $\mathcal{T}^c$ and $\mathcal{T}^\dagger$ was obtained for Γ. According to Section 2.1 part (4), the tree $\mathcal{T}^\dagger$ yields an admissible affinoid covering $\{X_v, X_e \mid \text{all } v, e\}$ of Ω. The result which makes counting possible is

Theorem 5.1. *The ramification points of the map $\Omega \to \Omega/\Gamma$ are contained in the union of the affinoids X_v corresponding to the vertices $[M_v]$ of $\mathcal{T}^\dagger$ with v a vertex of $\mathcal{T}^c$.*

Let $x = x_1 \in \Omega$ be a ramification point such that the order of the group $H := \{\gamma \in \Gamma \mid \gamma(x) = x\}$ is not divisible by p_K. Then H is cyclic and the other fixed point x_2 in $\mathbf{P}^1(K)$ of H also belongs to Ω.

Proof. Let $T_H \subset \mathcal{T}^\dagger$ be the subtree consisting of the vertices and edges which are invariant under $H := \{\gamma \in \Gamma \mid \gamma(x) = x\}$. We claim that T_H is a finite tree. Suppose that T_H is infinite then there exists $[M] \in T_H$ and there are infinitely many $\gamma \in \Gamma$ such that $[\gamma M] \in T_H$. Indeed, $\mathcal{T}^\dagger/\Gamma$ is a finite tree. Let G denote the stabilizer of $[M]$ in Γ. Then there are infinitely many $\gamma \in \Gamma$ such that $H \subset \gamma G \gamma^{-1}$. It follows that there are infinitely many $\gamma \in \Gamma$ with $\gamma H \gamma^{-1} \subset G$ and also infinitely many $\gamma \in \Gamma$ which commute with H. Now Lemma 4.6 yields the contradiction that x is a limit point.

Suppose that the order of H is not divisible by p_K. Then clearly, H is cyclic. If the second fixed point x_2 of H is a limit point, then this point determines a halfline in $\mathcal{T}^\dagger$ which is invariant under H. Since T_H is finite, this is not possible and $x_2 \in \Omega$.

(1) *The case $p_K = 0$ and $p_k \neq 2$.*

Let $x \in \Omega$ be a ramification point for Γ. Its stabilizer H is a maximal finite cyclic group subgroup of Γ with fixed points $x = x_1, x_2 \in \Omega$. Put $m = \#H$.

(1a) *Suppose that m is not a power of p_k and $m \neq 2$.* T_H is equal to the intersection of the axis of H in $\mathcal{BT}$ with $\mathcal{T}^\dagger$ and has the form $\{[M_1], \ldots, [M_s]\}$ with $s \geqslant 1$. If $[M] \in T_H$ is equal to a $[M_v]$ with v a vertex of $\mathcal{T}^c$, then v is not isomorphic to D_ℓ with ℓ a power of p_k. It follows that the reduction map $\mathrm{red}_{[M]} : \mathbf{P}^1(K) \to \mathbf{P}(M \otimes k)$ is injective on the set of the ramified points of $\mathbf{P}^1(K)$ for the group v. In particular, $\mathrm{red}_{[M]}(x_1) \neq \mathrm{red}_{[M]}(x_2)$. If $[M] \in T_H$ does not have the above form, then $[M]$ has at least three edges in the direction of vertices of the form $[M_v]$. It follows that $[M]$ lies in a segment $[[M_v], [M_{v'}]] \subset T_H$ with v, v' vertices of $\mathcal{T}^c$. Again $\mathrm{red}_{[M]}(x_1) \neq \mathrm{red}_{[M]}(x_2)$. For an edge $e = \{[M_i], [M_{i+1}]\}$ there is a $j \in \{1, 2\}$ with $\mathrm{red}_{[M_i]}(e) = \mathrm{red}_{[M_i]}(x_j)$ and $\mathrm{red}_{[M_{i+1}]}(e) = \mathrm{red}_{[M_{i+1}]}(x_j)$. For an extremal vertex $[M]$ of T_H one has $\mathrm{red}_{[M]}(e) \neq \mathrm{red}_{[M]}(x_1), \mathrm{red}_{[M]}(x_2)$ for every edge which does not belong to T_H. Now we conclude:

For $s = 1$ one has $[M_1] = [M_v]$ for some vertex v of $\mathcal{T}^c$ and $x_1, x_2 \in X_v$. For $s > 1$, one writes $[M_1] = [M_{v_1}]$ and $[M_s] = [M_{v_s}]$ with v_1, v_s vertices of $\mathcal{T}^c$ and (say) $x_1 \in X_{v_1}$, $x_2 \in X_{v_s}$.

(1b) *Suppose that* $m = 2$. As in case (1a), T_H lies on the axis of H and has the form $\{[N_1], \ldots, [N_s]\}$ with $s \geqslant 1$. An extremal vertex of T_H is again a $[M_v]$ for some $v \in \mathcal{T}^c$. A new possibility would be that $[N_1] = [M_v]$ for $v \cong D_\ell$ with ℓ a power of p_k. We compare this with 4.12 part (7). There are two other lattice classes invariant under v and hence under H, namely $[M_{-1}]$ and $[M_1]$. At least one of them does not belong to $\mathcal{T}^\dagger$. If neither belongs to $\mathcal{T}^\dagger$ (this is case (d)) then $x_1, x_2 \in X_v$. If one of the $\{[M_{\pm 1}]\}$ belongs to $\mathcal{T}^\dagger$ (this is case (e)), then $s > 1$ and one of the points x_1, x_2 belongs to X_v. The other fixed point of H lies in $X_{v'}$, where v' is the vertex of $\mathcal{T}^c$ satisfying $[M_{v'}] = [N_s]$.

(1c) *Suppose that* m *is a power of* p_k. Suppose that H is contained in at least two maximal finite subgroups v_1, v_2. Both groups are isomorphic to D_m and have separating invariant lattices $[M_1], [M_2] \in T_H$. After changing $[M_2]$ if necessary, one may suppose that the stabilizer of every $[M] \in [[M_1], [M_2]]$ is contained in v_1 or v_2. By Theorem 3.3, $v_1 *_H v_2$ is a subgroup of Γ. The points x_1, x_2 are limit points for this subgroup and we find a contradiction. Thus H is contained in a single maximal finite subgroup v and the points x_1, x_2 are lying in X_v (according to the tree of $D_{p_k^s}$ in Section 2).

(2) *The case* $p_K = 0$, $p_k = 2$.

Let $x \in \Omega$ be a ramification point for Γ. Its stabilizer H is a cyclic group of order $m = 2, 3$ or 2^s with $s > 1$. For $m = 3$, every maximal finite subgroup v of Γ, containing H is isomorphic to A_4. The proof of (1a) can be copied in this situation.

For $m = 2^s$ with $s > 1$, every maximal finite subgroup $v \supset H$ of Γ is isomorphic to D_{2^s}. As in (1c) one shows that H is contained in only one maximal finite subgroup v and that $x_1, x_2 \in X_v$.

For $m = 2$, we consider the axis $L \subset \mathcal{BT}$ of H and the finite collection V of all maximal finite subgroup v of Γ such that H is maximal cyclic in v. We may suppose that V consists of more than one element. For any two groups $v, v' \in V$ one has that $v *_H v'$ is a realizable amalgam. The proof of Theorem 3.5 shows that the lattice class $[M_v]$ does not lie on L. Moreover, the projections $\{\mathrm{pr}_L([M_v]) \mid v \in V\}$ have to be distinct. There are $v_1, v_2 \in V$ such that $\mathrm{pr}_L(v_1)$, $\mathrm{pr}_L(v_2)$ are extremal vertices of the set $\{\mathrm{pr}_L([M_v]) \mid v \in V\}$. Then (say) x_1 belongs to X_{v_1} and x_2 belongs to X_{v_2}.

(3) *The case* $p_K = p \geqslant 5$.

If the stabilizer H of $x \in \Omega$ has an order m not divisible by p, then the method of (1a) can be applied to prove that the two fixed points lie in affinoids X_v with $v \in \mathcal{T}^c$.

Suppose that the stabilizer H of $x \in \Omega$ contains an element of order p. If H is a maximal finite subgroup of Γ, then $H \cong B(n, m)$ with $m > 1$ since we have excluded $p = 2, 3$. Then $v = H$ is a vertex of $\mathcal{T}^c$. By the definition of $[M_v]$ one has that $\mathrm{red}_{[M_v]}\, x$ is different from $\mathrm{red}_{[M_v]}(e)$ for every edge of $[M_v]$ in $\mathcal{T}^\dagger$. It follows that $x \in X_v$.

If H is not a maximal finite subgroup, then a maximal finite subgroup $v \supset H$ of Γ must be isomorphic to $\mathrm{PGL}_2(\mathbf{F}_q)$ or $\mathrm{PSL}_2(\mathbf{F}_q)$. Again, $H \cong B(n, m)$ with $m > 1$ and moreover H is a maximal proper subgroup of v. The lattice classes of $\mathcal{BT}$, stabilized by H, form a half line (see 4.12 part (5)). Suppose that H is contained in another maximal finite subgroup v', then $[[M_v],[M_{v'}]]$ lies on this half line and we may suppose that for any $[M] \in [[M_v],[M_{v'}]]$, $[M] \neq [M_v],[M_{v'}]$ its stabilizer $\Gamma_{[M]} \supset H$ is not a maximal finite subgroup. If $\Gamma_{[M]} \neq H$, then $\Gamma_{[M]}$ lies in a third maximal finite subgroup v''. Also v'' lies on this halfline and H is a maximal proper subgroup of v, v', v''. This yields a contradiction and we conclude that $\Gamma_{[M]}$ must be H. Theorem 3.1 implies that $v *_H v'$ is a realizable amalgam. By 3.11 and $p \neq 2,3$ this is not possible. Thus v is the only maximal finite subgroup containing H. Then for every edge e of v, the group Γ_e is distinct from H. Thus $x \in X_v$.

(4) *The case* $p_K = p = 2,3$.

Let $x \in \Omega$ be a ramification point for Ω with stabilizer H in Γ. If H is not a p-group, then, as in (3) above, one can show that $x \in X_v$ for some vertex v of $\mathcal{T}^c$. If $H \cong C_p$ and H is contained in only one maximal finite subgroup v of Γ, then v is a vertex of $\mathcal{T}^c$ and $x \in X_v$. Any other p-group H is a maximal p-group and H is a vertex of $\mathcal{T}^c$. Let $[M_H]$ be the associated lattice class. By definition, the reduction map $\mathrm{red}: \Omega \to \overline{(\Omega, \mathcal{T}^\dagger)}$ has the property that $\mathrm{red}(x)$ lies on only one irreducible component, namely the one corresponding with $[M_H]$. Thus $x \in X_H$. □

Definition 5.2. Now we define the objects and numbers which will appear in the formulas for the number of branch points $\mathrm{br}(\Gamma)$ of Γ. Let $\mathcal{M}ax(i)$, $i = 2,3$, be the set of conjugacy classes of maximal finite subgroups $H \subset \Gamma$ such that $\mathrm{br}(H) = i$. Put $\max(i) = \#\mathcal{M}ax(i)$.

For $p_K = p = 2,3$ we consider $\mathcal{M}axp :=$ the set of conjugacy classes of the subgroups $H \subset \Gamma$ such that H is a maximal p-group and, moreover, H intersects at least two maximal finite, non-conjugated, subgroups $H_1, H_2 \subset \Gamma$ such that $H \cap H_1 = H \cap H_2 \cong C_p$. For $\alpha \in \mathcal{M}axp$, represented by the group $H \subset \Gamma$, one puts $d_\alpha := \#\{\beta \in \mathcal{M}ax(2) \mid \exists (H_1 \in \beta)\ H \cap H_1 \cong C_p\}$. We note that d_α is equal to the number of edges in $T^c := \mathcal{T}^c/\Gamma$ of the vertex $v \in T^c$ such that Γ_v belongs to α. Further we define $maxp := \sum_{\alpha \in \mathcal{M}axp}(d_\alpha - 1)$.

For $p_K = 0$, $\mathcal{M}axc$ denotes the set of conjugacy classes of maximal finite cyclic subgroups of Γ. For $p_K = p > 0$, $\mathcal{M}axc$ is the set of conjugacy classes of maximal finite cyclic subgroups $H \subset \Gamma$ such that $p \nmid \#H$.

For a class $\alpha \in \mathcal{M}axc$, represented by H, we define the integer $m_\alpha := \#\{\beta \in \mathcal{M}ax(2) \cup \mathcal{M}ax(3) \mid \exists (H_1 \in \beta)\ H \subset H_1\}$. Put $maxc := \sum_{\alpha \in \mathcal{M}axc,\, m_\alpha \neq 0}(m_\alpha - 1)$.

We note that $m_\alpha - 1$ gives the number of edges e in T^c such that the stabilizer Γ_e contains a maximal cyclic subgroup contained in the conjugacy class α. Further, *maxc* is equal the number of edges in T^c such that Γ_e is not a p-group.

For a finite group G acting on some space A, we will write $\mathrm{br}(G, A)$ for the number of branch points of the map $A \to A/G$. Moreover, we will write $\mathrm{br}(G)$ for $\mathrm{br}(G, \mathbf{P}^1(K))$.

Theorem 5.3. *Let $\Gamma \subset \mathrm{PGL}_2(K)$ be a finitely generated infinite discontinuous, indecomposable and ordinary group with $\Omega/\Gamma \cong \mathbf{P}^1_K$. We fix an embedding of $T^c := \mathcal{T}^c/\Gamma$ into $\mathcal{T}^c$. Then the number of branch points of Γ satisfies*

(1) $\mathrm{br}(\Gamma) = \sum_{v \text{ vertex of } T^c} \mathrm{br}(\Gamma_v) - \sum_{e \text{ edge of } T^c} \mathrm{br}(\Gamma_e)$.
(2) $\mathrm{br}(\Gamma) = 3 \cdot \mathit{max}(3) + 2 \cdot \mathit{max}(2) - \mathit{maxp} - 2 \cdot \mathit{maxc}$.
(3) $\mathrm{br}(\Gamma) = \mathit{max}(3) + \mathit{maxp} + 2$.

In particular, $\mathit{maxp} = 0$ if $p_K \neq 2, 3$ and $\mathrm{br}(\Gamma) = 2 + \#\{v \text{ vertex of } T^c\}$ if $p_K = 0$ or $p_K \geqslant 5$.

Proof. We note that formulas (2) and (3) use only the structure of Γ. We will first show that formula (1) implies the other two formulas.

For $p_K = 0$ one has $\mathrm{br}(\Gamma_v) = 3$. For $p_K = p > 0$ one has:

$$\mathrm{br}(\Gamma_v) = 3 \quad \text{if and only if} \quad p \nmid \#\Gamma_v,$$

$$\mathrm{br}(\Gamma_v) = 2 \quad \text{if and only if} \quad p \mid \#\Gamma_v \text{ and } \Gamma_v \text{ is not a } p\text{-group},$$

$$\mathrm{br}(\Gamma_v) = 1 \quad \text{if and only if} \quad \Gamma_v \text{ is a } p\text{-group (occurs only for } p = 2, 3).$$

Let N_1 (resp. $N_{e,1}$) be the number of vertices v (resp. edges e) in T^c such that Γ_v (resp. Γ_e) is a p-group. Then $\mathrm{br}(\Gamma) = 3 \cdot \mathit{max}(3) + 2 \cdot \mathit{max}(2) + N_1 - 2 \cdot \mathit{maxc} - N_{e,1}$. The vertices $v \in T^c$ with Γ_v a p-group, not involved in $\mathcal{M}\mathit{axp}$, are extremal vertices. Hence $N_1 + \mathit{maxp} = N_{e,1}$. This implies (2). Formula (3) is obtained by noting that $\mathit{max}(3) + \mathit{max}(2) + N_1 - \mathit{maxc} - N_{e,1} = 1$, since T^c is a tree.

Now we prove formula (1). By 5.1, the set of the ramification points of $\Omega \to \Omega/\Gamma$ is the disjoint union of the sets $\mathrm{Ram}(\Gamma_v, X_v)$ with v a vertex of $\mathcal{T}^c$, consisting of the ramification points for the groups Γ_v acting upon the affinoid set X_v attached to the vertex v. Then $\mathrm{br}(\Gamma) = \sum_v \mathrm{br}(\Gamma_v, X_v)$, where the sum is taken over the vertices of T^c. We recall that ψ_v is the reduction map $\mathbf{P}^1(K) \to \mathbf{P}(M_v \otimes k)$. The set $\mathrm{Ram}(\Gamma_v, \mathbf{P}^1(K))$ of the ramification points of Γ_v acting upon $\mathbf{P}^1(K)$ is the disjoint union of $\mathrm{Ram}(\Gamma_v, X_v)$ and the sets $\mathrm{Ram}(\Gamma_e, \mathrm{red}_v^{-1} \mathrm{red}_v(e))$, taking over the edges e of v in $\mathcal{T}^c$, consisting of the ramification points of Γ_e on the set $\mathrm{red}_v^{-1} \mathrm{red}_v(e)$. One obtains the formula

$$\mathrm{br}(\Gamma_v, X_v) = \mathrm{br}(\Gamma_v) - \sum_e \mathrm{br}\bigl(\Gamma_e, \mathrm{red}_v^{-1} \mathrm{red}_v(e)\bigr),$$

where the sum is taken over all edges e of v in T^c. Let H be a finite subgroup of Γ. Then formula (1) follows from

$$\sum_e \mathrm{br}(\Gamma_e) = \sum_{e, v_i} \mathrm{br}\bigl(\Gamma_e, \mathrm{red}_{v_i}^{-1} \mathrm{red}_{v_i} e\bigr), \tag{$*$}$$

where the first sum is taken over the edges e of T^c with $\Gamma_e = H$ and the second sum is taken over the same edges and the vertices v_i in T^c of those edges.

The verification of $(*)$ for $p_K = 0$, follows easily from the assumption Γ ordinary and $[M_v]$ is the separating lattice class if $v \cong D_\ell$ with ℓ a power of p_k.

For $p_K = p > 0$, all cases which do not involve a vertex v with Γ_v a p-group, follow by straightforward computation from the possibilities given by Corollary 3.11 and (e2), (e3) of Definition 4.3. The remaining cases are $p = 2, 3$, $H \cong C_p$ and there exists a vertex v of T^c with $H \subset v$ and v is a p-group. Let $e_i = \{v, v_i\}$, $i = 1, \ldots, s$, denote the edges of v in T^c. Every Γ_{v_i} is a dihedral group if $p = 2$ and is $\cong \mathrm{PSL}_2(\mathbf{F}_3)$ if $p = 3$. Now $\mathrm{br}(\Gamma_{e_i}) = 1$, $\mathrm{br}(\Gamma_{e_i}, \mathrm{red}_{v_i}^{-1} \mathrm{red}_{v_i} e_i) = 1$, $\mathrm{br}(\Gamma_{e_i}, \mathrm{red}_v^{-1} \mathrm{red}_v(e_i)) = 0$ for all i. This proves $(*)$. □

Remark 5.4. In [6, Theorem 1], Kato has given a list of finitely generated, infinite discontinuous subgroups $\Gamma \subset \mathrm{PGL}_2(K)$ with $\Omega/\Gamma \cong \mathbf{P}^1_K$ with $\mathrm{br}(\Gamma) = 3$ for the case $p_K = 0$. In particular, he shows that such groups only exist if $p_k \leqslant 5$. In the corollary below, we recover this part of his result.

Corollary 5.5. *Suppose $p_K = 0$. Let $\Gamma \subset \mathrm{PGL}_2(K)$ be a finitely generated, infinite, discontinuous group. Let v denote any maximal finite subgroup of Γ. Assume that every indecomposable component of Γ is ordinary. According to* 4.2 *this is equivalent to*:

(1) *If $p_k > 2$ and $p_k \mid \#v$, then v is a dihedral group, and*
(2) *If $p_k = 2$, then $v \in \{A_4, D_{2^s}$ with $s \geqslant 1\}$.*

Then $\mathrm{br}(\Gamma) \geqslant 4$. *Moreover* $\mathrm{br}(\Gamma) = 4$ *holds only for the following situations*:

(a) *Γ is indecomposable and T^c consists of a single edge.*
(b) *Γ is a free amalgam of two finite* (*non-trivial*) *cyclic subgroups.*

Proof. If Γ is indecomposable and infinite, then $\mathrm{br}(\Gamma) = 2 +$ the number of vertices of T^c and thus $\geqslant 4$. For a (non-trivial) finite subgroup $H \subset \mathrm{PGL}_2(K)$ one has $\mathrm{br}(H) = 2$ or 3. Hence, if Γ is infinite and decomposable, then $\mathrm{br}(\Gamma) \geqslant 4$. Therefore $\mathrm{br}(\Gamma) = 4$ only occurs if Γ is a free amalgam of two finite non-trivial cyclic groups. □

Corollary 5.6. *Let $p_K = p > 0$. A chosen embedding of $T^c := \mathcal{T}^c/\Gamma$ into $\mathcal{T}^c$ makes T^c into a tree of groups. For vertices v and edges e of T^c one writes Γ_v and Γ_e for the corresponding groups.*

(1) *If Γ_v is of Borel type, not a p-group, then v is an extremal vertex of T^c.*
(2) *Suppose that v is not an extremal vertex and $p \mid \#\Gamma_v$. Let $e_i = \{v_i, v\}$, $i = 1, \ldots, s$, denote the edges of v. The only possibilities are*
   (a) *Γ_v is not of Borel type, $s = 2$ and* (*after renumbering*) *$p \mid \#\Gamma_{e_1}$, $p \nmid \#\Gamma_{e_2}$, and Γ_{v_1} is of Borel type.*
   (b) *$p = 2, 3$, Γ_v is a p-group and all $\Gamma_{e_i} \cong C_p$.*
(3) *If p divides the order of every vertex group of T^c and* (2b) *holds for no vertex of T^c which has at least two edges, then T^c has two extremal vertices and at most four vertices.*
(4) *Suppose that Γ_v is of Borel type for every vertex v of T^c. Then Γ is isomorphic to $B(n_1, m) *_{C_m} B(n_2, m)$ with $m > 1$ and $n_1, n_2 > 1$.*

Proof. (1) follows at once from part (4) of Theorem 4.11.

(2) Suppose that Γ_v is not of Borel type. Since $[M_v]$ is separating and $\mathrm{br}(\Gamma_v) = 2$, one has $s = 2$. Moreover, one of the branch points is wildly ramified and the other is tamely ramified, see Proposition 2.1. Hence $p \mid \#\Gamma_{e_1}$ and $p \nmid \#\Gamma_{e_2}$. By 3.11 and (e2), (e3) of Definition 4.3, Γ_{v_1} is of Borel type (and can even be a p-group).

Suppose that Γ_v is of Borel type. Then $\Gamma_v = B(n, 1)$ and $p = 2, 3$ and all $\Gamma_{e_i} \cong C_p$, by 3.11.

(3) By assumption, every vertex v of T^c, which is not extremal, satisfies $p \mid \#\Gamma_v$, Γ_v is a p-group and not of Borel type by (1). Hence v has two edges and T^c has two extremal vertices. Let $v_1, \ldots, v_s$ denote the vertices of T^c and let $\{v_i, v_{i+1}\}$, $i = 1, \ldots, s-1$, be the edges. Then $s \leqslant 4$ since $p \mid \#\Gamma_{v_i}$ and Γ_{v_i} is not a p-group and even not of Borel type for $i = 2, \ldots, s-1$.

(4) Follows at once from 3.11 and (3) above. □

Remark 5.7. In [2, Proposition 4.6], Cornelissen et al. have determined all finitely generated discontinuous subgroups $\Gamma \subset \mathrm{PGL}_2(K)$ with $\mathrm{br}(\Gamma) = 2$ for the case where $p_K = p > 0$. The proposition below recovers part of their result.

Proposition 5.8. *Let $p_K = p > 0$ and let Γ be infinite. We assume that* $\mathrm{br}(\Gamma) = 2$. *If Γ is indecomposable, then we fix an embedding of $T^c := \mathcal{T}^c/\Gamma$ into $\mathcal{T}^c$. Then the following holds*:

- (i) *If Γ is indecomposable, then $p|\#\Gamma_v$ for all vertices v of T^c and one of the following statements holds*:
  - (a) $\Gamma \cong B(n_1, m) *_{C_m} B(n_2, m)$ *with* $m, n_1, n_2 > 1$.
  - (b) *There is precisely one vertex v of T^c such that Γ_v is not of Borel type.*
  - (c) *Precisely two vertices v_1, v_2 of T^c have groups Γ_v which are not of Borel type.*
- (ii) *If Γ is decomposable, then Γ is a free amalgam of two p-groups.*

Proof. (i) Suppose that Γ is indecomposable, then $\mathrm{br}(\Gamma) = \mathit{max}(3) + \mathit{maxp} + 2$. Since $\mathrm{br}(\Gamma) = 2$, one must have that $\mathit{max}(3) = \mathit{maxp} = 0$. In particular, for every vertex v of T^c, the order of Γ_v is divisible by p and if v is not an extremal vertex then Γ_v is not a p-group. One applies now Corollary 5.6.

(ii) Γ is the free amalgam of two discontinuous groups $\Gamma_1, \Gamma_2 \subset \Gamma$ with $\mathrm{br}(\Gamma_1) = \mathrm{br}(\Gamma_2) = 1$. Both Γ_1 and Γ_2 are clearly finite p-groups. □

Remark 5.9 (Discontinuous groups which are not finitely generated). Let $p_K = p > 0$. There are many natural examples of discontinuous groups $\Gamma \subset \mathrm{PGL}_2(K)$ such that Ω/Γ is isomorphic to $\mathbf{P}^1_K \setminus S$, where S is a finite, non-empty set. E.g., let $A = \mathbf{F}_q[t] \subset K$ with $|t| > 1$. Then $\Gamma = \mathrm{PGL}_2(A)$ is a discontinuous group such that $\Omega/\Gamma \cong \mathbf{A}^1_K$ and in particular Γ is not finitely generated.

We will indicate how one can extend the results of Sections 4 and 5 to discontinuous groups Γ as above. As before, one associates a tree $\mathcal{T}$ to Ω. In particular, the definition of indecomposable group Γ still makes sense and one can decompose Γ as a free product $\Gamma_1 * \cdots * \Gamma_s$ of indecomposable groups of the same type. For an indecomposable Γ as above, one defines the tree $\mathcal{T}^c$ as follows. The *vertices v of $\mathcal{T}^c$* are the following subgroups of Γ:

(i) Maximal finite subgroups of Γ.
(ii) For $p = 2, 3$, subgroups $H \subset \Gamma$, $H \cong C_p$ satisfying (v2) of Definition 4.3.
(iii) Infinite stabilizers Γ_y of points $y \in \mathbf{P}^1(K)$.

We note that an infinite stabilizer Γ_y is isomorphic to a semi-direct product $N \rtimes A$, where $N \subset K$ is an infinite discontinuous group and $A \subset K^*$ is a finite group. An *edge of the tree $\mathcal{T}^c$* is a pair of vertices $v_1, v_2 \in \mathcal{T}^c$ for which there exist vertices $\bar{v}_1, \bar{v}_2 \in \mathcal{T}$ such that one of the following holds:

(i) The stabilizer $\Gamma_{\bar{v}_i}$ is a non-cyclic subgroup of Γ_{v_i} for $i = 1, 2$. Furthermore, the stabilizer of any vertex between $\bar{v}_1$ and $\bar{v}_2$ is contained in Γ_{v_1} or Γ_{v_2} and, moreover, $\Gamma_{v_1} \cap \Gamma_{v_2} \ncong C_p$.
(ii) All elements of either Γ_{v_1} or Γ_{v_2} have order p and $\Gamma_{v_1} \cap \Gamma_{v_2} \cong C_p$.

To the vertices of the tree $\mathcal{T}^c$ one cannot always associate lattice classes. If the stabiliser Γ_v of a vertex $v \in \mathcal{T}^c$ is infinite, then one associates to v the line L_v in V that corresponds to the unique point $y \in \mathbf{P}^1_K$ that is stabilized by the group Γ_v. If the group Γ_v is finite, then one associates to v a lattice class $[M_v]$ as before. The lattice classes contained in the convex hull of all lattices classes $[M_v]$ and all lines L_v with $v \in \mathcal{T}^c$ a vertex, define again a locally finite tree $\mathcal{T}^\dagger$.

We note that the tree $\mathcal{T}^c$ is not locally finite. Indeed, if the stabilizer Γ_v of a vertex $v \in \mathcal{T}^c$ is infinite, then the vertex v is contained in infinitely many edges. The tree $\mathcal{T}^c/\Gamma$, however, is still finite. One can verify that the proofs of 4.9, 4.10, et cetera, remain valid, mutatis mutandis.

For the number of branch points $\mathrm{br}(\Gamma)$ of $\Omega \to \Omega/\Gamma$ one finds again $\mathrm{br}(\Gamma) = \sum_i \mathrm{br}(\Gamma_i)$ holds. Let $E(\Gamma)$ denote the number of ends of the tree $\mathcal{T}/\Gamma$. Then $E(\Gamma)$ is the cardinality of S. One has $E(\Gamma) = \sum_{i=1}^{s} E(\Gamma_i)$.

For an indecomposable Γ we will give the formula for $\mathrm{br}(\Gamma)$. We fix an embedding of $\mathcal{T}^c/\Gamma$ into $\mathcal{T}^c$ and let $T^{nB} \subset \mathcal{T}^c/\Gamma$ consist of the vertices $v \in \mathcal{T}^c/\Gamma$ such that Γ_v is not contained in a Borel subgroup of $\mathrm{PGL}_2(K)$. Then T^{nB} is either empty or a disjoint union of subtrees of $\mathcal{T}^c/\Gamma$.

Suppose that $T^{nB} = \emptyset$. Then $\mathcal{T}^c/\Gamma$ consists of at most two vertices. If $\mathcal{T}^c/\Gamma$ consists of a single vertex, then we do not define a group Γ^{nB}. In that case Γ is an infinite discontinuous subgroup of a Borel subgroup of $\mathrm{PGL}_2(K)$ and $E(\Gamma) = 1$. Moreover, $\mathrm{br}(\Gamma) = 0$ if all elements of Γ have order equal to p. Otherwise, $\mathrm{br}(\Gamma) = 1$. If $\mathcal{T}^c/\Gamma$ consist of a single edge e, then we put $\Gamma^{nB} := \Gamma_e$.

Suppose that $T^{nB} \neq \emptyset$. Then Γ^{nB} denotes the group generated by the groups Γ_v with $v \in T^{nB}$. For the Γ, for which $\Gamma^{nB} \subset \Gamma$ is well-defined, one has $\mathrm{br}(\Gamma) = \mathrm{br}(\Gamma^{nB}) - E(\Gamma)$. We remark that Γ^{nB} is finitely generated and indecomposable and thus $\mathrm{br}(\Gamma^{nB})$ is given by Theorem 5.3.

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